Towards Three-Dimensional Conformal Probability

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Conference in honor of Vincent Rivasseau
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1. The Brydges-Mitter-Scoppola (BMS) model in 3D
2. The p-adic hierarchical model

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Should correspond to critical scaling limit of long-range three-dimensional Ising model with ferromagnetic interactions $J_{\mathbf{x},\mathbf{y}} \sim |\mathbf{x} - \mathbf{y}|^{-(d+\sigma)}$, $d = 3$, $\sigma = \frac{3+\epsilon}{2}$.

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where $[\phi] = \frac{3-\epsilon}{4}$ is the scaling dimension of the field. Let $\mu_{C_{-\infty}}$ be the centered Gaussian measure with covariance $C_{-\infty}$.

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Fix rescaling ratio $L > 1$, integer.

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Given bare ansatz $(g_r, \mu_r)_{r\in\mathbb{Z}}$, one has well defined probability measures $d\nu_{r,s}(\phi)$ whose Radon-Nikodym derivative with respect to $d\mu_{C_r}(\phi)$ is proportional to

$$\exp\left(-\int_{\mathbb{R}^3} \rho_{\mathrm{IR},s}(x)\left\{g_r :\phi^4:(x) + \mu_r :\phi^2:(x)\right\} d^3x\right)$$

with Wick ordering relative to μ_{C_r}.

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The scale invariant BMS measure should be the weak limit $\nu_\phi = \lim_{r\to-\infty} \lim_{s\to\infty} \nu_{r,s}$ for a well chosen bare ansatz which mimics the scaling limit of a critical theory on the unit lattice.

Conjecture 1:

Set $[\phi] = \frac{3-\epsilon}{4}$ for ϵ positive and small.
Then there exists a nonempty interval $I \subset (0, \infty)$ and a function $\mu_{\rm c} : I \to \mathbb{R}$ such that for all $g \in I$, if one sets $g_r = L^{-r(3-4[\phi])} g$ and $\mu_r = L^{-r(3-2[\phi])} \mu_{\rm c}(g)$, the weak limit ν_ϕ exists and is non-Gaussian, translation-invariant, $O(3)$-invariant, OS positive, and scale-invariant with exponent $[\phi]$, i.e., $\lambda^{[\phi]} \phi(\lambda \cdot) \stackrel{dd}{=} \phi(\cdot)$ for all $\lambda > 0$.
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Measure constructed on finite torus by Mitter (~ 2004) using fixed point obtained by BMS, CMP 2003.

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The moments

$$S_n(f_1, \ldots, f_n) = \langle \phi(f_1) \cdots \phi(f_n) \rangle = \int_{S'(\mathbb{R}^3)} \phi(f_1) \cdots \phi(f_n) d\mu(\phi)$$

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Conj. 3 is a precise formulation of predictions in "Conformal invariance in the long-range Ising model" by Paulos, Rychkov, van Rees and Zan, arXiv:1509.00008[hep-th] $- >$ Higher-dimensional conformal bootstrap.

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One-to-one correspondence: $f \in \mathcal{M}(\mathbb{R}^3) \ \leftrightarrow$ hyperbolic isometry of the bulk $\mathbb{B}^4$ or $\mathbb{H}^4$.

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Let p be an integer > 1 (in fact a prime number).

Let $\mathbb{L}_k$, $k \in \mathbb{Z}$, be the set of boxes $\prod_{i=1}^{d} \left[a_i p^k, (a_i + 1)p^k\right)$ for $a_1, \ldots, a_d \in \mathbb{N}$. The cubes in $\mathbb{L}_k$ form a partition of the octant $[0, \infty)^d$.

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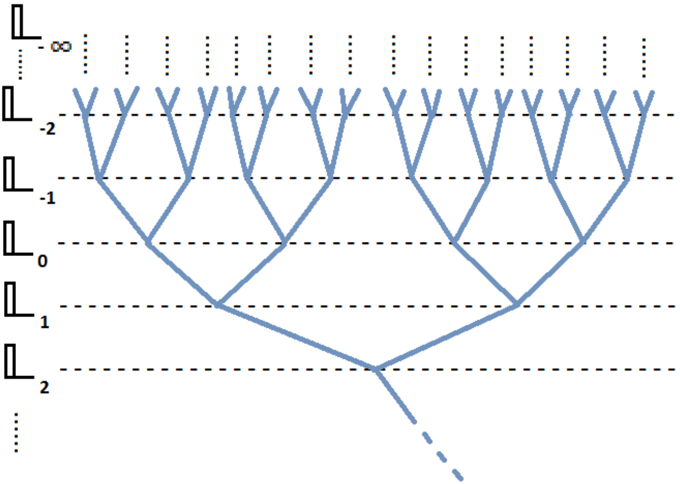
Then $\mathbb{T} = \cup_{k \in \mathbb{Z}} \mathbb{L}_k$ naturally has the structure of a doubly infinite tree organized in layers or generations $\mathbb{L}_k$:



Picture for $d = 1$, $p = 2$

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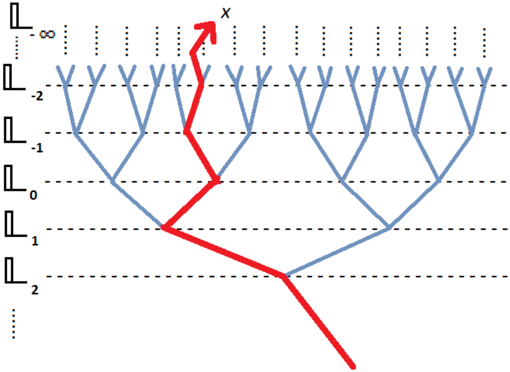
More precisely, this is the set of upward paths in the tree.



A path representing some $x \in \mathbb{Q}_p^d$

A point $x \in \mathbb{Q}_p^d$ encoded by sequence $(a_n)_{n\in\mathbb{Z}}$, $a_n \in \{0, 1, \ldots, p-1\}^d$.
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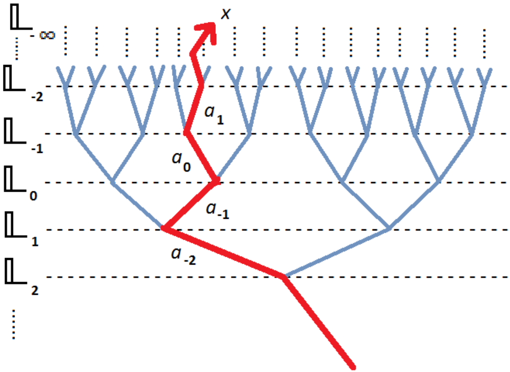

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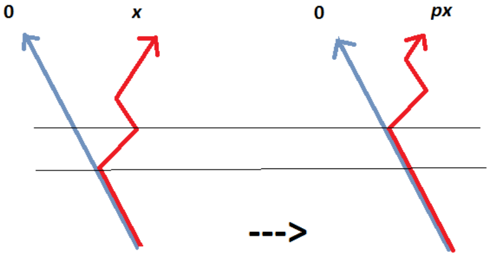

Likewise $p^{-1}x$ is downward shift and so on for defining $p^k x$, $k \in \mathbb{Z}$.

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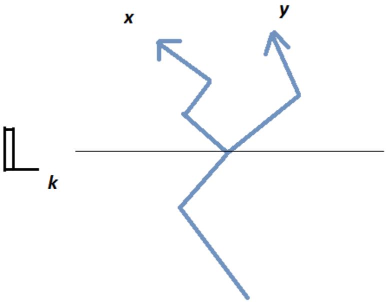

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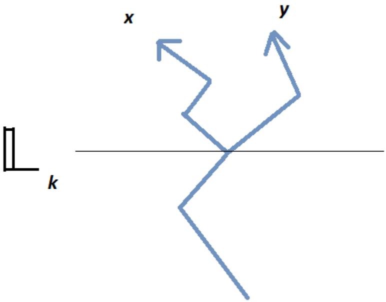

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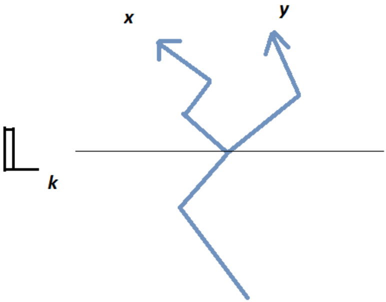

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$$|px| = p^{-1}|x|$$

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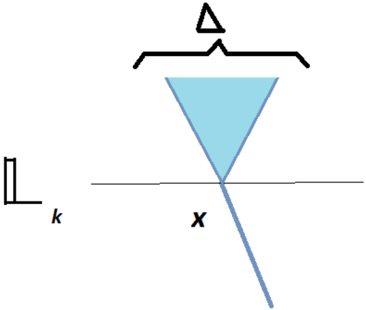

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Construction: take product of uniform probability measures on $(\{0, 1, \ldots, p-1\}^d)^{\mathbb{N}}$ for $\overline{B}(0, 1)$ and similarly for other balls of radius 1, then collate.

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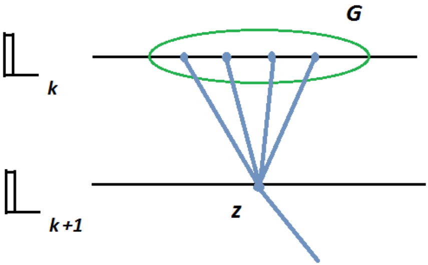

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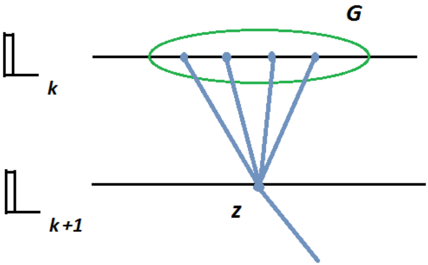

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only formal since ϕ not defined pointwise. Need random distributions.

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Topology generated by the set of all seminorms.

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Probability theory on $S'(\mathbb{Q}_p^d)$ is very nice!

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6. $S'(\mathbb{Q}_p^d) \times S'(\mathbb{Q}_p^d) \simeq S'(\mathbb{Q}_p^d)$ so same tools work for joint law of pair of distributional random fields, e.g., $(\phi, N[\phi^2])$.

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Gaussian measures are scaled copies of each other.
If law of $\phi(\cdot)$ is μ_{C_0}, then law of $L^{-r[\phi]}\phi(L^r \cdot)$ is μ_{C_r}.

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Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{g_r :\phi^4:_{C_r}(x) + \mu_r :\phi^2:_{C_r}(x)\} d^3x$$

and define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi)$$

Let $\phi_{r,s}$ random variable in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define square field $N_r[\phi_{r,s}^2]$ which is deterministic $S'(\mathbb{Q}_p^3)$-valued function of $\phi_{r,s}$ given by

$$N_r[\phi_{r,s}^2](j) = Z_2^r \int_{\mathbb{Q}_p^3} \{Y_2 :\phi_{r,s}^2 :_{C_r} (x) - Y_0 L^{-2r[\phi]}\}\, j(x)\, d^3x$$

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Our main result concerns the limit law of the pair $(\phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ regardless of the order of limits.

Will need approximate fixed point coupling

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})}$$

8) Results

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2. $\langle\phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} < 0$ i.e., ϕ is non-Gaussian. Here $\mathbf{1}_{\mathbb{Z}_p^3}$ is the indicator function of $\overline{B}(0,1)$.

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3. $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} = 1$.

Mixed correlations satisfy in sense of distributions

$$\langle \phi(L^{-k}x_1) \cdots \phi(L^{-k}x_n) N[\phi^2](L^{-k}y_1) \cdots N[\phi^2](L^{-k}y_m) \rangle$$

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$[\phi^2]-2[\phi]=\frac{1}{3}\epsilon+o(\epsilon)$ as expected in Euclidean BMS model.

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The law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$ is independent of g: universality

Theorem 2: A.A.-Chandra-Guadagni 2013

$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under action of $p^{\mathbb{Z}}$ instead of just $L^{\mathbb{Z}}$. Moreover, $\mu(g)$ and $[\phi^2]$ independent of RG step L.

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Note that $3-2[\phi^2] = 3-\frac{1}{3}\epsilon + o(\epsilon)$ $->$ still $L^{1,\mathrm{loc}}$!

Theorem 3: A.A., May 2015

Let ψ_i denote ϕ or $N[\phi^2]$. Then for every mixed correlation $\exists$ smooth function $\langle\psi_1(z_1)\cdots\psi_n(z_n)\rangle$ on $(\mathbb{Q}_p^3)^n\backslash\mathrm{Diag}$ which is locally integrable (even on Diag) such that

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Namely, $\nu_{\phi\times\phi^2}$ satisfies the DPC property or the analogue of Conj. 2.

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The AdS bulk is the tree $\mathbb{T}$ with the graph distance. Analogue of hyperbolic metric.

Mumford-Manin-Drinfeld Cross-Ratio Lemma

$$CR(x_1, x_2, x_3, x_4) := \frac{|x_1 - x_3|\ |x_2 - x_4|}{|x_1 - x_4|\ |x_2 - x_3|} = p^{-\delta(x_1 \to x_2; x_3 \to x_4)} ,$$

where $\delta(x_1 \to x_2; x_3 \to x_4)$ is the number of common edges in bi-infinite paths $x_1 \to x_2$ and $x_3 \to x_4$, counted positively if orientations agree and negatively otherwise.

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$$CR(x_1, x_2, x_3, x_4) := \frac{|x_1 - x_3|\ |x_2 - x_4|}{|x_1 - x_4|\ |x_2 - x_3|} = p^{-\delta(x_1 \to x_2; x_3 \to x_4)}\ ,$$

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Using the cross-ratio lemma, one can again establish a one-to-one correspondence: $f \in \mathcal{M}(\mathbb{Q}_p^3) \leftrightarrow$ hyperbolic isometry of the bulk $\mathbb{T}$.

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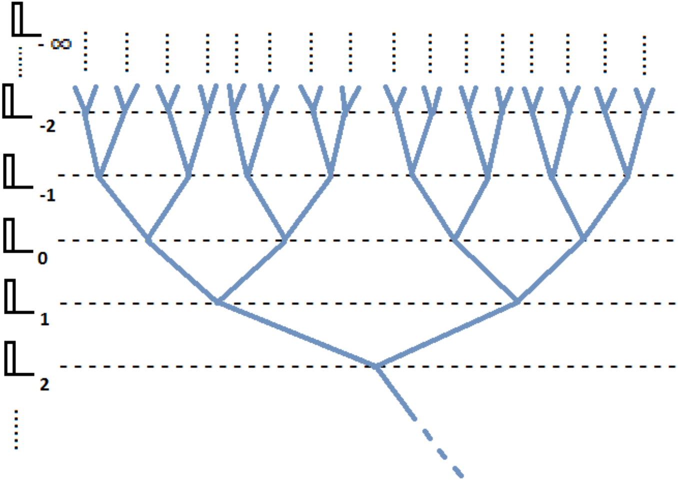
Rigorous RG for space-dependent couplings in ACG 2013 $->$ space-dependent cut-offs $->$ Conj. 3. by showing equivalence of usual upper half-space cut-off with conformal ball cut-off.



The tree again

10) The last slide

10) The last slide

Cher Vincent,

10) The last slide

Cher Vincent,

Je te souhaite encore beaucoup d'années,

10) The last slide

Cher Vincent,

Je te souhaite encore beaucoup d'années,
et encore beaucoup d'étudiants.

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Joyeux Anniversaire!