On a power counting theorem for a $p^{2a}\varphi^4$ tensor field theory

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based on arXiv:1507.00590

Constructive Field Theory: from Condensed Matter to Quantum Gravity.
in honor of Vincent Rivasseau

Institut Henri Poincaré
Paris, France
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Outline

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1. Introduction

2. The $p^{2a}\varphi^4$ model
   - The action
   - Progrator and Feynman graphs
   - Amplitudes
   - Multi-scale analysis

3. Conclusion

Quantum Geometry by Colored Tensor Models

- Tensor Models (TM) of rank D: Tools for randomizing geometry in dimension D.
- Case $D = 2$: Matrix Models and QG in 2D.
- Basic building blocks $(D-1)$-simplexes & Interaction forms a D-simplex; [Ambjorn et al. '91, Boulatov-Ooguri '92]; Group Field Theory [Oriti, '05–].

- '10 Gurau's $1/N$ expansion for colored TM [Gurau, AHP, '11]

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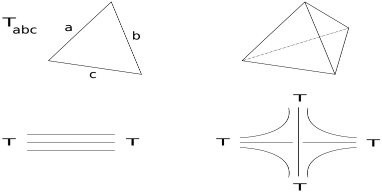

- triangulate mo' regular objects (pseudo-manifolds) [Gurau, CMP '11]
- $\exists\, 1/N$ and Leading graphs triangulate only spheres in any D [Gurau, AHP '11; Bonzom, Gurau, Rivasseau '15]
- have computable phase transition, and critical exponent [Bonzom, Gurau, Riello, Rivasseau, NPB, '11];
- At the effective level, they define renormalizable field theories called TFTs or TGFTs [BG & Rivasseau, '11; Carrozza, Oriti, Rivasseau '12; Samary & Vignes-Tourneret '12; BG, '13—; Avohou, Benedetti, Lahoche, Krajewski, Martini, Toriumi].

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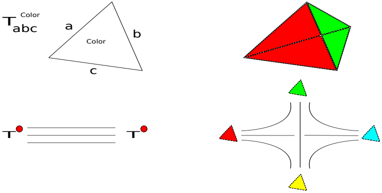

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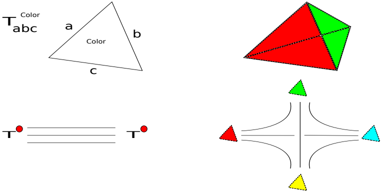

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Escaping from the branch polymer phase

• TM is a rich framework.

• Explore exotic models in order to resum more contributions and escape the BP phase: the Enhanced TM program [Enhancing non-melonic triangulations: A tensor model mixing melonic and planar maps, Bonzom, Delepouve & Rivasseau, NPB 2015]

• Caveat: It becomes difficult to identify a proper geometrical interpretation.

But not in the QFT setting: they belong to the theory space!

• Today, in the field theory setting, I will present a class of models which enhance terms which are ordinarily suppressed in power counting.

Outline

Tensorial field theory on $U(1)^d$

- $\phi : U(1)^d \to \mathbb{C}$, and its Fourier modes $\phi_{\mathbf{P}}$, with $\mathbf{P} = (p_1, p_2, \ldots, p_d)$, $p_k \in \mathbb{Z}$.
- The action:

$$S[\bar{\phi}, \phi] = \sum_{\mathbf{P}} (\bar{\phi}_{\mathbf{P}} \cdot \big(\sum_{i=1}^{d} p_i^2 \big) \cdot \phi_{\mathbf{P}}) \; + \; \mu \sum_{\mathbf{P}} \bar{\phi}_{\mathbf{P}} \phi_{\mathbf{P}} \; + \; S^{\text{int}} [\bar{\phi}, \phi] \,. \tag{1}$$

A new tensorial field theory on $U(1)^d$

- Interaction part: NONLOCAL !

Given $a \in (0, \infty)$,

$$S^{\text{int}}[\bar\phi, \phi] = \frac{\lambda}{2}\,\mathrm{Tr}_4(\phi^4) + \frac{\eta}{2}\,\mathrm{Tr}_4(p^{2a}\,\phi^4)\,,$$

$$\mathrm{Tr}_4(\phi^4) := \mathrm{Tr}_{4;1}(\phi^4) + \mathrm{Sym}(1 \to 2 \to \cdots \to d)\,,$$

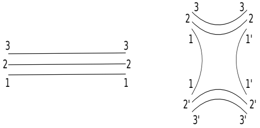
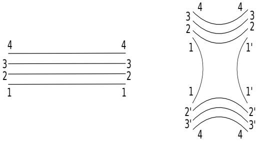
$$\mathrm{Tr}_4(p^{2a}\phi^4) := \mathrm{Tr}_{4;1}(p_1^{2a}\,\phi^4) + \mathrm{Sym}(1 \to 2 \to \cdots \to d)\,, \qquad (2)$$

and, in rank $d = 3$ and $d = 4$,

$$\mathrm{Tr}_{4;1}(\phi^4) = \sum_{p_i, p'_i \in \mathbb{Z}} \phi_{123}\,\bar\phi_{1'23}\,\phi_{1'2'3'}\,\bar\phi_{12'3'}\,,$$

$$\mathrm{Tr}_{4;1}(p_1^{2a}\,\phi^4) = \sum_{p_i, p'_i \in \mathbb{Z}} \left(p_1^{2a} + {p'}_1^{2a}\right)\phi_{123}\,\bar\phi_{1'23}\,\phi_{1'2'3'}\,\bar\phi_{12'3'}\,, \qquad (3)$$

- Feynman graphs: in rank $d = 3$ (left) and $d = 4$ (right)

Amplitudes and slice decomposition

• Graph amplitudes: $\mathcal{G}$ with set $\mathcal{V}$ of vertices (with $V = |\mathcal{V}|$) and set $\mathcal{L}$ of propagator lines (with $L = |\mathcal{L}|$)

$$A_{\mathcal{G}} = \sum_{p_{v;s}} \prod_{l \in \mathcal{L}} C_l(\{p_{v(l)}\}; \{p'_{v'(l)}\}) \prod_{v \in \mathcal{V}} (-V_{4;v}(\{p_{v;s}\})) . \qquad (4)$$

• Slice decomp.:

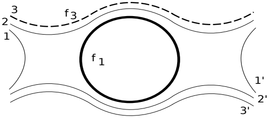
$$\tilde{C}(\{p_s\}) = \int_0^\infty d\alpha \; e^{-\alpha(\sum_s p_s^2 + \mu)} = \sum_{i=0}^{\infty} C_i(\{p_s\}) ,$$

$$C_i(\{p_s\}) = \int_{M^{-2(i+1)}}^{M^{-2i}} d\alpha e^{-\alpha(\sum_s p_s^2 + \mu)} \leq K M^{-2i} e^{-\delta M^{-i}(\sum_s |p_s| + \mu)} , \qquad (5)$$

Multi-scale analysis and Optimization

$$A_{\mathcal{G}} = \sum_{\boldsymbol{\mu}} A_{\mathcal{G};\boldsymbol{\mu}}\,, \qquad A_{\mathcal{G};\boldsymbol{\mu}} = \sum_{p_{v;s}} \prod_{l\in\mathcal{L}} C_{i_l}(\{p_{v(l)}\};\{p'_{v'(l)}\}) \prod_{v\in\mathcal{V}} (-V_{4;v}(\{p_{v;s}\}))\,,$$

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa(\lambda) \prod_{l\in\mathcal{L}} M^{-2i_l} \sum_{p_{f_s}} \prod_{f_s\in\mathcal{F}_{\text{int}}} e^{-\delta(\sum_{l\in f_s} M^{-i_l})|p_{f_s}|} \prod_{s=1}^{d} \prod_{v_s\in\mathcal{V}_s} [1+\tilde{\eta}(\varepsilon\,\tilde{p}^{\,2a})_{v_s}]\,,$$



$$\varepsilon_{v_s f_{s'}} = \begin{cases} 1, & \text{if } s=s' \text{ and if } \; v_s \in f_s, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

• Given f, among the lines $l\in f$, use the line l_f with $i_{l_f} = \min_{l\in f} i_l = i_f$, which will generate the lowest factor M^{i_f}. Call i_f, the face scale index of f.

• To optimize the products of the vertex kernels: we must target, in each factor of the product of the vertex kernels, the term p_f generating after summation a product of $M^{i_f(2a\alpha+1)}$ with the largest possible power.

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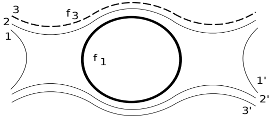

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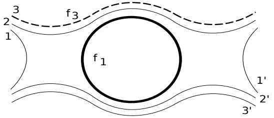

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Multi-scale analysis: Optimization

- Investigate the combinatorics of the ε matrix.

$i_{f_{(1),1}} \geq i_{f_{(1),2}} \geq \dots,\ i_{f_{(2),1}} \geq i_{f_{(2),2}} \geq \dots$, etc...

	$V^{(1)}_1$	$V^{(1)}_2$	$\dots$	$V^{(1)}_{k_1}$	$V^{(2)}_1$	$\dots$	$V^{(2)}_{k_2}$	$\dots$
$f_{(1),1}$	1	1	0	0	x	x	x	x
$f_{(1),2}$	0	1	0	0	x	x	x	x
$f_{(1),3}$	0	1	0	1	x	x	x	x
$f_{(2),1}$	x	x	x	x	1	0	0	x
$f_{(2),2}$	x	x	x	x	0	1	1	x
$f_{(2),3}$	x	x	x	x	0	0	0	x

(7)

- Start with the face $f_{s;1}$, and count $\varrho_{f_{s;1}} = \sum_l \varepsilon_{v_{s,l} f_{s.1}}$, i.e. the number of vertices $v_{s;l}$ such that $\varepsilon_{v_{s;l} f_{s;1}} = 1$.

Define

$$\varrho(\mathcal{G}) = \sum_s \sum_{f_{s;k}} \varrho_{f_{s;k}} . \tag{8}$$

- $\varrho(\mathcal{G}) \leq V(\mathcal{G})$

Power counting theorem

Then

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa_2 \prod_{l\in\mathcal{L}} M^{-2i_l} \prod_{f_s\in\mathcal{F}_{\rm int}} M^{i_{f_s}(2a\varrho_{f_s}+1)}\,, \qquad (9)$$

Power counting

Let $A_{\mathcal{G};\boldsymbol{\mu}}$ be the amplitude associated with the graph $\mathcal{G}$ of the $p^{2a}\varphi^4_d$-model in the multi-scale index $\boldsymbol{\mu}$, then there exists a constant κ depending on the graph such that

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq \kappa \prod_{(i,k)\in\mathbb{N}^2} M^{\omega_{\rm d}(G^i_k)}\,, \qquad (10)$$

where G^i_k are quasi-local subgraphs and

$$\omega_{\rm d}(G^i_k) = -2L(G^i_k) + F_{\rm int}(G^i_k) + 2a\varrho(G^i_k)\,. \qquad (11)$$

- $a \to 0$, one recovers the usual power counting for tensorial field theory over $U(1)$.
- $2a\varrho(G^i_k)$ enhances the divergence degree.

Properties

- Non-melonic graphs might diverge and can even dominate melonic ones.

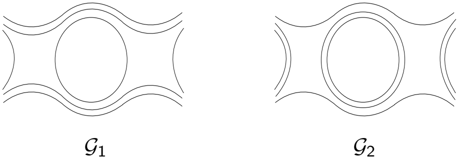

Figure: Two rank 3 4-point graphs: $\mathcal{G}_1$ is a not a melon and $\mathcal{G}_2$ is.

Example: Non-melonic 4-point graph $\mathcal{G}_1$ with (superficial) degree of divergence:

$$\omega_{\rm d}(\mathcal{G}_1) = -2 \times 2 + 1 + 2a \times 2 = 4a - 3 \qquad (12)$$

which is strictly positive, whenever $a > \frac{3}{4}$.
The 4-point melonic graph $\mathcal{G}_2$ in the same figure, one finds

$$\omega_{\rm d}(\mathcal{G}_2) = -2 \times 2 + 2 = -2 < 0$$

which implies a convergent amplitude.

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Conclusion

• A new type of Tensorial Field Theory incorporating Laplacian on the interaction.
• These terms appear naturally in the expansion of the Functional Renormalization Group Equations for TFTs.
• Result: The ordinary suppressed terms become enhanced.

• Future investigations:

- How is this useful to the continuum limit?
- Towards new classes of (just) renormalizable models?

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