The Coulomb gas in two dimensions

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60th Birthday of Vincent Rivasseau
Paris, Nov 2015

Charles Augustin de Coulomb 1736 – 1806

Les Sciences sont des monumens consacrés au bien public; chaque citoyen leur doit un tribut proportionné à sès talens. Tandis que les grands hommes, portés au sommet de l'édifice, tracent et élèvent les étages supérieurs, les artistes ordinaires répandus dans les étages inférieurs, ou cachés dans l'obscurité des fondemens, doivent seulement chercher à perfectionner ce que des mains plus habiles ont créé.

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- $Z_\Lambda(\beta, z) = \lim_{m^2\to 0} \sum_{\omega\in\Omega} \frac{z_\omega^n}{n_\omega!} e^{-\beta H_\Lambda(\omega)}$.

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- $\rho_\eta(a-b)$ is $e^{-\beta\delta(\text{Energy})}$.

Debye and Hückel 1923

Theorem (Wei-Shih Yang 1987)

For the grand canonical Coulomb system on $\mathbb{R}^2$ *with* β *small depending in an implicit way on* z*,* $\rho_\eta(a,b)$ *decays exponentially to* $\rho_\eta(a)\rho_\eta(b)$ *as* $|a-b| \to \infty$*.*

Open problems: (1) implicit hypothesis on z; (2) free boundary conditions to relate to Fröhlich results; (3) extend range of β (4) $\beta <$ KT transition for lattice model?

Theorem (Fröhlich-Spencer 1981)

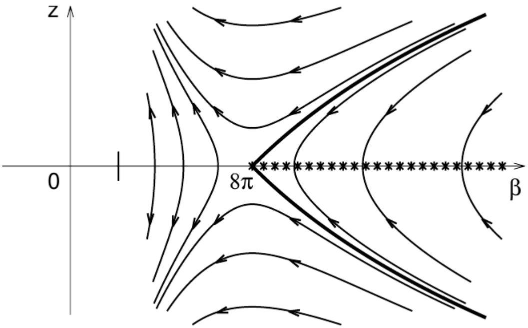
For the Coulomb system on $\mathbb{Z}^2$ *exponential screening of fractional charges does not hold for* β *large.*

Fröhlich, J. (1976). Classical and quantum statistical mechanics in one and two dimensions: two-component Yukawa- and Coulomb systems.
Comm. Math. Phys., 47(3):233–268

Yang, W.-S. (1987). Debye screening for two-dimensional Coulomb systems at high temperatures.
J. Statist. Phys., 49:1–32

Fröhlich, J. and Spencer, T. (1981). The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas.
Comm. Math. Phys., 81(4):527–602

KT Picture



β_{eff} := where trajectories cross horizontal axis.

$\rho(a-b)$ decays as $|a-b|^{-2\kappa}$ for $\eta \in (0, \frac{1}{2}]$ and $\beta_{\text{eff}} \geq 8\pi$.

$\kappa = \frac{\beta_{\text{eff}}}{4\pi}\eta^2$ with log corrections for $\beta_{\text{eff}} = 8\pi$.

Berezinskiĭ, V. L. (1970). Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group. I. Classical systems.
Ž. Èksper. Teoret. Fiz., 59:907–920

Kosterlitz, M. and Thouless, D. J. (1973). Ordering, metastability and phase transitions in two-dimensions.
J. Phys. C, 6:1181–1203

Kosterlitz, J. M. (1974). The critical properties of the two-dimensional xy model.
Journal of Physics C: Solid State Physics, 7(6):1046

Theorem (Pierluigi Falco, 2013)

KT picture, including differential equations for trajectories, holds with explicit log corrections to κ for $\beta_{\mathrm{eff}} = 8\pi$ and z small.

The FS result was improved to $\beta_{\mathrm{eff}} > 8\pi$.

Falco, P. (2012). Kosterlitz-Thouless transition line for the two dimensional Coulomb gas.
Comm. Math. Phys., 312(2):559–609

Falco, P. (2013). Critical exponents of the two dimensional coulomb gas at the Berezinskii-Kosterlitz-Thouless transition.
http://arxiv.org/abs/1311.2237

Marchetti, D. H. U. and Klein, A. (1991). Power-law falloff in two-dimensional Coulomb gases at inverse temperature $\beta > 8\pi$.
J. Statist. Phys., 64(1-2):135–162

Sine–Gordon transformation

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$$\rho_\eta(a - b) = \lim_{\Lambda \to \infty} \langle e_{a,\eta} e_{b,-\eta} \rangle_\Lambda.$$

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$$\Omega_R(J,\Lambda) = \mathbb{E}\left[e^{V_{,0}(J,\varphi)}\right]$$

$$V_{,0}(J,\varphi), = \underbrace{\sum_{x,\mu} \frac{s}{2} (\partial^\mu \varphi_x)^2}_{\text{counterterm}} + \sum_{x,\sigma} (z e_{x,\sigma} + J_{x,\sigma} e_{x,\sigma\eta})$$

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Dropping terms that cancel with denominator

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constructs $\Omega_R(J, \varphi)$

Begin definition of RG:

$$\begin{tikzcd}
\Omega_j \arrow[rr, "\mathbb{E}_j"] & & \Omega_{j+1} \\
(s_j, z_j, Z_j, \bar{Z}_j, K_j) \arrow[u] \arrow[rr, "\mathrm{RG}"'] & & (s_{j+1}, z_{j+1}, Z_{j+1}, \bar{Z}_{j+1}, K_{j+1}) \arrow[u]
\end{tikzcd}$$

- bulk coupling constants s_j, z_j
- Real valued observable coupling constants $Z_j, \bar{Z}_j$
- K_j in Banach space

Definition of vertical arrows: step 1

Given $(s_j, z_j, Z_j, \bar{Z}_j) \in \mathbb{R}^4$, define functions of $\Phi = (J, \varphi)$.

$$V_{0,j}(\Phi, B) = \sum_{x \in B, \mu} \frac{s_j}{2} (\partial^\mu \varphi_x)^2 + L^{-2j} \sum_{x \in B, \sigma = \pm} z_j e_{x,\sigma}$$

$$V_{1,j}(\Phi, B) = L^{-2j} \sum_{x \in B, \sigma = \pm} Z_j J_{x,\sigma} e_{x,\sigma\eta} + L^{-2j} \sum_{x \in B, \sigma = \pm} \bar{Z}_j J_{x,\sigma} e_{x,\bar{\eta}}, \qquad \bar{\eta} = 1 - \eta.$$

$$V_{,j}(\Phi, B) = V_{0,j}(\Phi, B) + V_{1,j}(\Phi, B)$$

Definition of vertical arrows: step 2

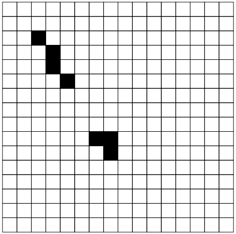
Let

$$U_j(\Phi, B) = V_{,j}(\Phi, B) + W_{,j}(\Phi, B)$$

where $W_{,j}(\Phi, B)$ is another explicit function of $\Phi = (J, \varphi)$ defined by $(s_j, z_j, Z_j, \bar{Z}_j)$.

It is given by a LARGE formula obtained from second order perturbation theory.

Definition of vertical arrows



Given $K_j : X \mapsto$ function of $(\varphi_x, J_x)_{x \in X^\square}$

Ω_j is expressed in terms of (U_j, K_j)

using

$$\Omega_j(\Phi, \Lambda) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y).$$

K_j is there to include the remainder after second order perturbation theory.

Summary, so far

$$\begin{tikzcd}
\Omega_j \arrow[rr, "\mathbb{E}_j"] & & \Omega_{j+1} \\
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- U_j determined by coupling constants $(s_j, z_j, Z_j, \bar{Z}_j)$
- $\Omega_j(\Phi, \Lambda) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y)$,

Theorem ($\exists$ RG)

For all j such that (s_j, z_j) is small, K_j is $O(s_j, z_j)^3$ uniformly in j, and (s_j, z_j) follows the KT picture:

$$s_{j+1} \approx s_j - a z_j^2, \qquad z_{j+1} \approx L^2 e^{-\frac{\alpha^2}{2} Var(\zeta_j)} [z_j - b s_j z_j]$$

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Theorem (Best choice of s)

For $\alpha^2 = 8\pi$ and $z_0 = z$ small, there is a unique $s_0 = s_0(z)$ such that (s_j, z_j, K_j) is in the domain of RG for all scales j and (s_R, z_R, K_R) tends to zero in the infinite volume limit $R \to \infty$.

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$z_R \to 0$ means there are no monopoles at macroscopic scales.
$s_R \to 0$ means that dipoles, quadrupoles, ... absorbed into gaussian $\alpha\varphi$.

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So if

$$\alpha^2 = 8\pi, \qquad \text{KT point}$$

then z_j is marginal.

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To have $\alpha^2 = 8\pi$, by the definition $\alpha^2 = (1 - s_0(z))\beta$, get formula for KT critical (β, z) line

$$\beta = \frac{8\pi}{1 - s_0(z)}$$

Calculation of $\rho(a, b)$

After R steps Λ becomes a single block so that

$$\Omega_R(\Phi, \Lambda) = e^{U_R(\Phi,\Lambda)} + K_R(\Phi, \Lambda).$$

Put this into

$$\rho_\eta(x, y) = \frac{1}{\Omega_R(\Phi, \Lambda)} \frac{\partial^2 \Omega_R(\Phi, \Lambda)}{\partial J_x \partial J_y}\Big|_{J=0}.$$

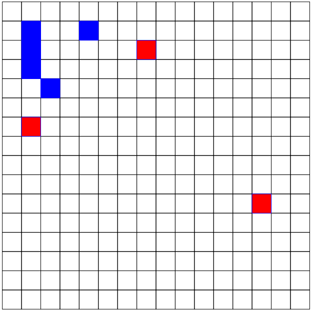
In the infinite volume limit $R \to \infty$, K_R becomes zero and makes no contribution.

$\rho(a, b)$ is completely determined by the double derivative of $W_{,R}$ and the $(s, z, Z, \bar{Z})$ flow.

Recall the magenta arrow

$$\begin{tikzcd}
\Omega_j \arrow[r, "\mathbb{E}_j"] & \Omega_{j+1} \\
(U_j, K_j) \arrow[u] \arrow[r, "\mathrm{RG}"'] & (U_{j+1}, K_{j+1}) \arrow[u]
\end{tikzcd}$$

Part of definition of $(U_j, U_{j+1}, K_j) \mapsto K_{j+1}$



$\Omega_j =$

$$\sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y).$$

In each small block expand

$$\varphi = \varphi' + \zeta_j$$

$$e^{U_j(\varphi' + \zeta_j)} = e^{U_{j+1}(\varphi')} + \text{difference}.$$

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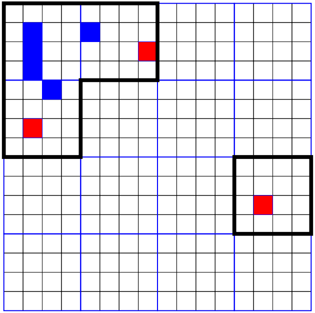

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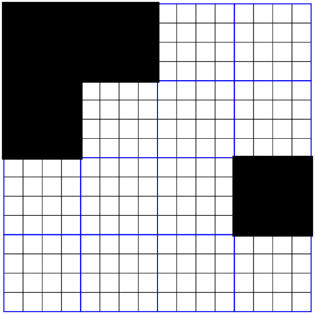

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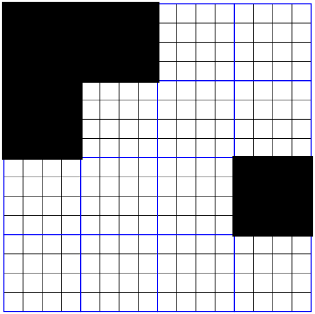

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Finite range: expectation factors over connected components.

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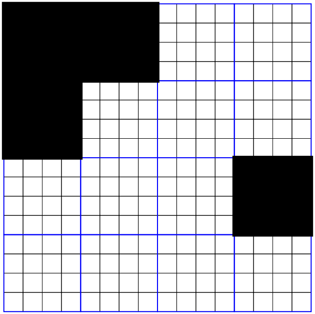

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Finite range: expectation factors over connected components.

For a connected union X of big blocks,

$$K_{j+1}(X) = \mathbb{E}_j\Big(\text{sum over ways to fill } X\Big).$$

Linear part on small sets

$X \mapsto K_{j+1}(X)$ is a power series in K_j. The linear term in this series is

$$X \mapsto \sum_{Y:\overline{Y}=X, |Y|_j \le 2^d} \mathbb{E} K_j(Y)$$

when coupling constants are zero.

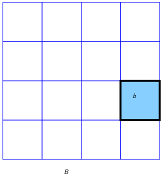
By very general arguments, the theorems above reduce to showing that this part of K_{j+1} is contractive as a function of K_j.

Brydges, D. C. (2009). Lectures on the renormalisation group.
In *Statistical Mechanics*, volume 16 of *IAS/Park City Math. Ser.*, pages 7–93. Amer. Math. Soc., Providence, RI

Brydges, D. and Yau, H.-T. (1990). Grad ϕ perturbations of massless Gaussian fields.
Comm. Math. Phys., 129(2):351–392

Example

Consider a scale $j+1$ block B.



The linearisation of $K_j \mapsto K_{j+1}(B)$ is

$$\sum_{b \in \mathcal{B}_j(B)} K_j(b)$$

For a generic K_j, it would expand by L^2 because there are L^2 little blocks b inside B.

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- Vertical arrows:

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$$

- Vertical arrows:

$$\Omega_j(\Phi, \Lambda) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y), \qquad \Phi = (J, \varphi)$$

- U_j determined by coupling constants $(s_j, z_j, Z_j, \bar{Z}_j)$:

$$
\begin{tikzcd}
\Omega_j \arrow[rr, "\mathbb{E}_j"] & & \Omega_{j+1} \\
& (\tilde{U}_j, \tilde{K}_j) \arrow[ul] \arrow[dr] & \\
(U_j, K_j) \arrow[uu] \arrow[ur] \arrow[rr, dashed, "\mathrm{RG}"'] & & (U_{j+1}, K_{j+1}) \arrow[uu]
\end{tikzcd}
$$

- Vertical arrows:

$$\Omega_j(\Phi, \Lambda) = \sum_{X \in \mathcal{P}_j} e^{U_j(\Phi, \Lambda \setminus X)} \prod_{Y \in \mathcal{C}_j(X)} K_j(\Phi, Y), \qquad \Phi = (J, \varphi)$$

- U_j determined by coupling constants $(s_j, z_j, Z_j, \bar{Z}_j)$:

Pierluigi's list of open problems

In a talk given in 2011 (video on IAS website),

- Analyticity in z inside the dipole phase and Borel summability on the KT line.
- Extension to other models discussed by Fröhlich–Spencer? XY, Villain, discrete Gaussian, Z_n-clock, and solid-on-solid.
- Equivalence of Coulomb gas and other 2D probabilitic models at criticality: Ashkin–Teller, six-vertex, Q-state and antiferromagnetic Potts model, $O(n)$-models including self-avoiding walk.

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