A Topological Recursion for Tensor Models

Conference in honor of Vincent Rivasseau

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Outline

Matrix models for 2d quantum gravity

Integrals of matrices with Feynman graphs = poly-angulations of surfaces.

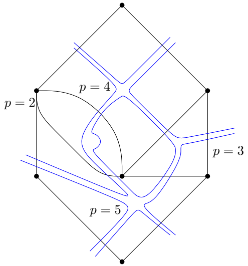


Figure: Example of a poly-angulation and its dual Feynman Graph.

Curvature concentrates at vertices of the p-angulation $\rightarrow$ faces of the graph

Observables of the model:

$$\{\mathrm{tr}(M^p) | p \in \mathbb{N}\}$$

Represent boundary states of the triangulated surfaces.

"Transition amplitudes" = numbers of triangulations of surfaces with corresponding boundaries.

Higher dimensions? **Tensor models**. Generalize the techniques of matrix models to tensor models.

Topological recursion

Generating functions of observables:

$$\forall (g,n) \quad \text{s.t. } 2g-2+n \geq -2, \quad W_n^g(x_1,\ldots,x_n) = \sum_{p_i \geq 0} \frac{\langle \prod_i \mathrm{tr}(M^{p_i}) \rangle_c^g}{\prod_i x_i^{p_i+1}}$$

Loop equations:

$$W_{n+1}^{g-1}(x,x,x_I) + \sum_{\substack{0\leq h\leq g \\ J\subseteq I}} W_{1+|J|}^{h}(x,x_J)W_{1+|I-J|}^{g-h}(x,x_{|I-J|})$$
$$+\sum_{i\in I} \frac{\partial}{\partial x_i} \frac{W_n^g(x,x_2,\ldots,\hat{x}_i,\ldots,x_n) - W(x_2,\ldots,x_n)}{(x-x_i)^2}$$
$$+ V(x)W_n^g(x,x_I) + P_n^g(x;x_2,\ldots,x_n) = 0.$$

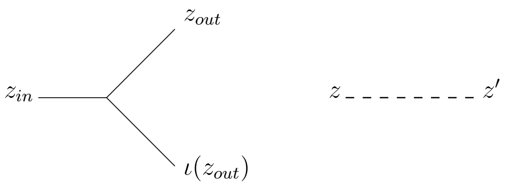
Solution of the loop equations

Compute W_1^0 and W_2^0.
The form of W_1^0 tells us about a function $x : \Sigma \to \mathbb{C} \setminus \bigcup_i \gamma_i$.
Define $\omega_n^g = W_n^g dx_1 \dots dx_n$ and you solve the loop equations by the following recurrence formula:

$$\omega_n^g(z_1,\dots,z_n) = \sum_{p_i} \underset{z\to p_i}{Res} K(z,z_1)\Big[\omega_{n+1}^{g-1}(z,\iota(z),z_2,\dots,z_n) + \sum_{\substack{0\le h\le g\\ J\subseteq I}}' \omega_{1+|J|}^h(z,z_2,\dots,z_n)\cdot\omega_{1+|I-J|}^{g-h}(\iota(z),z_2,\dots,z_n)\Big]. \quad (1)$$

All very classic now.

A combinatorial representation of solution.



$Res_{z_{out}\to p_i}\ K(z_{out}, z_{in})$ $\qquad$ $\omega_2^0(z, z')$

Figure: Building blocks of the Topological Recursion Graphs.

$\to g = 0$ graphs are trees. Adding loops on these trees $\nearrow g$.

The tensor model case.

After some work one shows the simplest interacting tensor model (remember the talk of Joseph!) reformulates,

$$Z[\alpha, N] = \int_{f, H_N^d} \prod_{c=1}^{d} dM_c e^{-\frac{N}{2}\sum_{c=1}^{d} \mathrm{tr}(M_c^2)} e^{-\mathrm{tr}\log_2\left[\mathbb{1}^{\otimes d} - \frac{\alpha^p}{N^{\frac{d-2}{2}}}\sum_{c=1}^{d}\mathcal{M}_c\right]}.$$

with

$$\mathcal{M}_c = \mathbb{1}^{\otimes(c-1)} \otimes M_c \otimes \mathbb{1}^{\otimes(d-c)}.$$

plenty of matrices M_c. **We focus here on** $d = 4n + 2$ as this implies the following slides.

Loop equations

Plenty of matrices: $W^g_n \to W^g_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^6$

General loop equations: notational nightmare, but let us write a part of it...

$$\sum_{\substack{g \geq h \geq 0 \\ \mathbf{q}+\mathbf{r}=\mathbf{k} | \mathbf{q},\mathbf{r},\mathbf{k} \in \mathbb{N}^{d=6}}} W^h_{e_1+\mathbf{q}}(x, x_{\mathbf{q}}) W^{g-h}_{e_1+\mathbf{r}}(x, x_{\mathbf{r}}) + W^{g-1}_{2e_1+\mathbf{k}}(x, x, x_{\mathbf{k}})$$

$$= \text{Some multi-linear operator on the } W^h_{\mathbf{q}}$$

$$\text{s.t. } 2h - 2 + |\mathbf{q}| < 2g - 2 + |\mathbf{k}|$$

This multi-linear operator does basically two operations:

1. construct combinations of derivatives of $W^g_{\mathbf{k}-e_1}$.
2. Taylor expand the generating function at ∞ in some variables and select one coefficient of this Taylor expansion.

We can infer enough analytical properties of $W^g_{\mathbf{k}}$ for the next result.

Colored Blobbed Topological Recursion

It has colors, it has a funny "blobbed" name, it has trees decorated with loops hidden in the graphs. It has all things Vincent enjoys!

Theorem

$$\omega^g_{\mathbf{k}} = \sum_{\Gamma \in \mathfrak{G}^g_{\mathbf{k}}} \frac{\varpi^0_\Gamma(z_{\mathbf{k}})}{|Aut(\Gamma)|}$$

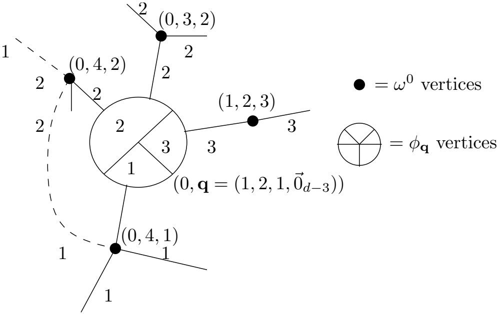
where $\mathfrak{G}^g_{\mathbf{k}} = \bigsqcup_{A,B} \mathfrak{G}^g_{\mathbf{k}}(A,B)$ is a set of graph, A, B are d-uplets $(A_i), (B_i)$ of subsets of $[\![1, k_i]\!]$ with $A_i \sqcup B_i = [\![1, k_i]\!]$.

$\varpi^0_\Gamma(z_{\mathbf{k}})$ is a weight associated to each graph $\Gamma \in \mathfrak{G}^g_{\mathbf{k}}(A,B)$.

Colored Blobbed Topological Recursion

First Promise: It has colors everywhere!

Let us look at one example of a graph $\Gamma \in \mathfrak{G}^2_{\mathbf{k}=(4,2,1,\vec{0}_{d-3})}(A, B)$,



A is such that $|A_1| = 1$, $|A_2| = |A_3| = 0$, B is such that $|B_1| = 3$, $|B_2| = 2$, $|B_3| = 1$.

Colored Blobbed Topological Recursion

Second promise: it has trees decorated with loops hidden in it!

The weight of the graphs compute from local weights associated to ω^0 vertices, $\phi_{\mathbf{k}}$ vertices, and bi-colored (dashed) edges and some pairing of these local weights.
But what are these local weights?

Colored Blobbed Topological Recursion

The secret for ω^0:
each ω^0 vertex comes with a bunch of labels (h, n, c). c is its color. h its genus. n its valency. One has $2h - 2 + n > 0$.

To each ω^0 vertex with these labels one associates a local weight $\omega^{h,0}_{n,c}(z_1, \dots, z_n)$. Indeed one has, for each $c[\![1, d]\!]$

$$\omega^{h,0}_{n,c}(z_1, \dots, z_n) = \sum_{\pm 1} \underset{z \to \pm 1}{Res} K(z, z_1) \Big[\omega^{h-1,0}_{n,c}(z, \iota(z), z_2, \dots, z_n)$$

$$+ \sum'_{\substack{0 \le h' \le h \\ J \subseteq I = [\![2,n]\!]}} \omega^{h',0}_{1+|J|,c}(z, J) \cdot \omega^{h-h',0}_{1+|I-J|,c}(\iota(z), z_{I-J}) \Big].$$

This is the same formula than before! $\Rightarrow$ expands on trees decorated with loops with the same rule than the usual topological recursion.

Colored Blobbed Topological Recursion

And the ϕ's?

Usual Topological Recursion: two initial conditions ω_1^0 and ω_2^0 one needs to compute by hand (need some "physical" input here).

Here infinite number of "initial conditions" $=$ the ϕ's. Practically one can write them as integral of some functions constructed from the potential of the model. This really comes from the the tensors variables.
There probably exists a recursive formula to compute them in the case of our 1-cut by color multi-matrix model. But we are deriving it, so no results at the moment.

To do list:

1. Compute the ϕ's: in progress...
2. Re-interpret tensor models observables in terms of moduli spaces intersection numbers: in progress... $\Rightarrow$ generalizes Givental decomposition.
3. Generalize to any tensor models, any dimensions? Some (very) vague ideas.
4. Use this framework to compute new scaling limit? Some (very) vague ideas.

Conclusion

Happy Birthday Vincent!