Complex Bosonic Many-body Models

Tadeusz Balaban
Rutgers University

Joel Feldman
University of British Columbia

Horst Knörrer, Eugene Trubowitz
ETH-Zürich

(P)reprints: http://www.math.ubc.ca/~feldman/
http://www.math.ubc.ca/~feldman/bec/

Abstract

I will discuss the current status of an ongoing program, with T. Balaban, H. Knoerrer and E. Trubowitz, whose long-term goal is the, mathematically rigorous, construction of a standard model of a gas of bosons.

The Physical Setting

Consider a gas of bosons.

- Each particle has a kinetic energy. For simplicity, let's take the corresponding quantum mechanical observable to be $-\jmath_0 \Delta$.
- The particles interact with each other through a translationally invariant, exponentially decaying, strictly positive definite two-body potential, $v(\mathbf{x}, \mathbf{y})$.
- The system is in the thermodynamic equilibrium given by the grand canonical ensemble with temperature $T \geq 0$ and chemical potential μ.
- I will concentrate on the partition function

$$Z = \mathrm{Tr}\, e^{-\frac{1}{kT}(H-\mu N)}$$

with $T > 0$, where H is the Hamiltonian and N is the number operator.

The Physics of Interest

Formally,

$$\mathrm{Tr}\, e^{-\frac{1}{kT}(H-\mu N)} = \int \prod_{\substack{\mathbf{x}\in\mathbb{R}^3 \\ 0<\tau\le\frac{1}{kT}}} \frac{d\psi_\tau(\mathbf{x})^* \wedge d\psi_\tau(\mathbf{x})}{2\pi i}\ e^{\mathcal{A}(\psi^*,\psi)}$$

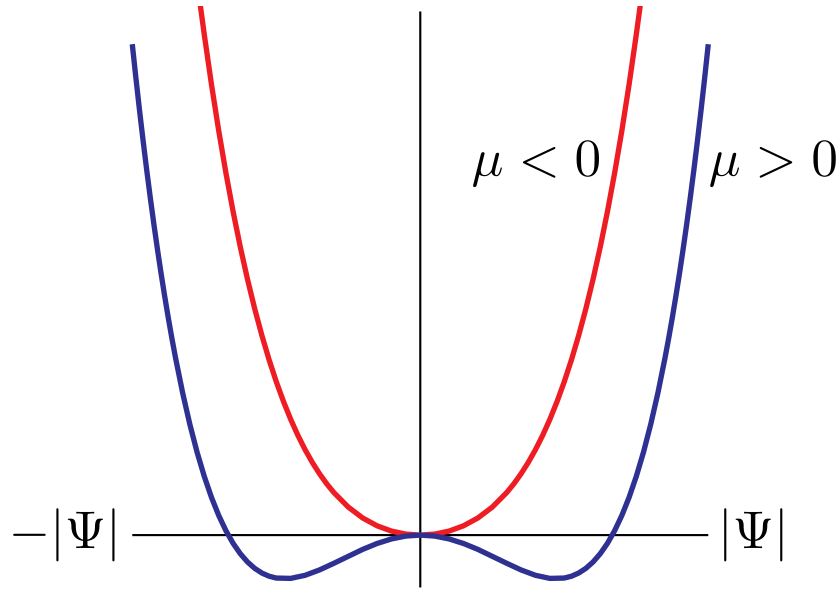
where the action

$$\left.\begin{aligned}\mathcal{A} = &\int_0^{\frac{1}{kT}} d\tau \int_{\mathbb{R}^3} d^3\mathbf{x}\quad \psi_\tau(\mathbf{x})^* \frac{\partial}{\partial\tau}\psi_\tau(\mathbf{x}) \\ &- \int_0^{\frac{1}{kT}} d\tau \iint d\mathbf{x}d\mathbf{y}\, \psi_\tau(\mathbf{x})^*(-\jmath_0\Delta\psi_\tau)(\mathbf{y}) \\ &+ \mu \int_0^{\frac{1}{kT}} d\tau \int d\mathbf{x}\quad \psi_\tau(\mathbf{x})^*\psi_\tau(\mathbf{x})\end{aligned}\right\}\text{propagator} = \frac{1}{ik_0 - \jmath_0\mathbf{k}^2 + \mu}$$

$$- \frac{1}{2} \int_0^{\frac{1}{kT}} d\tau \iint d\mathbf{x}d\mathbf{y}\, \psi_\tau(\mathbf{x})^*\psi_\tau(\mathbf{x})\, v(\mathbf{x},\mathbf{y})\, \psi_\tau(\mathbf{y})^*\psi_\tau(\mathbf{y})$$

If $\psi_\tau(\mathbf{x}) = \Psi \in \mathbb{C}$ is a constant, independent of τ and $\mathbf{x}$, the action $\mathcal{A}(\psi^*, \psi)$ simplifies to minus the integral over τ and $\mathbf{x}$ of the "naive effective potential"

$$\tfrac{1}{2}\hat{v}(0)|\Psi|^4 - \mu|\Psi|^2$$



where $\hat{v}(0) = \int d\mathbf{y}\, v(\mathbf{x}, \mathbf{y})$.

The minimum of this effective potential is

- nondegenerate at the point $\Psi = 0$ when $\mu < 0$ and
- degenerate along the circle $|\Psi| = \sqrt{\frac{\mu}{\hat{v}(0)}}$ when $\mu > 0$.

This suggests that for $\mu < \mu_{\rm crit}$, $\langle\psi(\mathbf{x})\rangle = 0$, just as you would expect from conservation of particle number. But for $\mu > \mu_{\rm crit}$, $\langle\psi(\mathbf{x})\rangle = \Psi$ for some complex number of modulus $|\Psi| \approx \sqrt{\frac{\mu}{\hat{v}(0)}} \neq 0$ and its precise value (i.e. which argument it has) will depend on the limiting process used to define the model. So we have to be very careful about how we define the model.

The Construction Plan – A Rigorous Starting Point

To carefully define the left hand side of

$$\mathrm{Tr}\, e^{-\frac{1}{kT}(H-\mu N)} = \int \prod_{\substack{\mathbf{x}\in\mathbb{R}^3 \\ 0<\tau\le\frac{1}{kT}}} \frac{d\psi_\tau(\mathbf{x})^*\wedge d\psi_\tau(\mathbf{x})}{2\pi i}\; e^{\mathcal{A}(\psi^*,\psi)}$$

you take a limit of obviously well–defined approximations. One way to get a (pretty) obviously well–defined approximation is to replace space $\mathbb{R}^3$ by a finite number of points, say $X = \mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3$. To get a (pretty) obviously well–defined approximation to the right hand side, replace "time", $[0, \frac{1}{kT}]$, by a finite number of points too.

Theorem (Functional Integral) *Suppose that* $\mathrm{R}(\varepsilon), \mathrm{r}(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$ *at suitable rates. For each fixed finite* X*,*

$$\mathrm{Tr}\ e^{-\frac{1}{kT}\,(H-\mu N)} = \lim_{\varepsilon\to 0} \int \prod_{\substack{\tau\in\varepsilon\mathbb{Z}\cap(0,\frac{1}{kT}] \\ \mathbf{x}\in X}} \frac{d\psi_\tau(\mathbf{x})^*\wedge d\psi_\tau(\mathbf{x})}{2\pi\imath}\ e^{\mathcal{A}_\varepsilon(\psi^*,\psi)}\ \chi(\psi^*,\psi) \tag{1}$$

with the convention that $\psi_0 = \psi_{\frac{1}{kT}}$ *and*

$$\mathcal{A}_\varepsilon(\psi^*,\psi) \approx \mathcal{A}(\psi^*,\psi) \quad \textit{when } \varepsilon \textit{ is small}$$

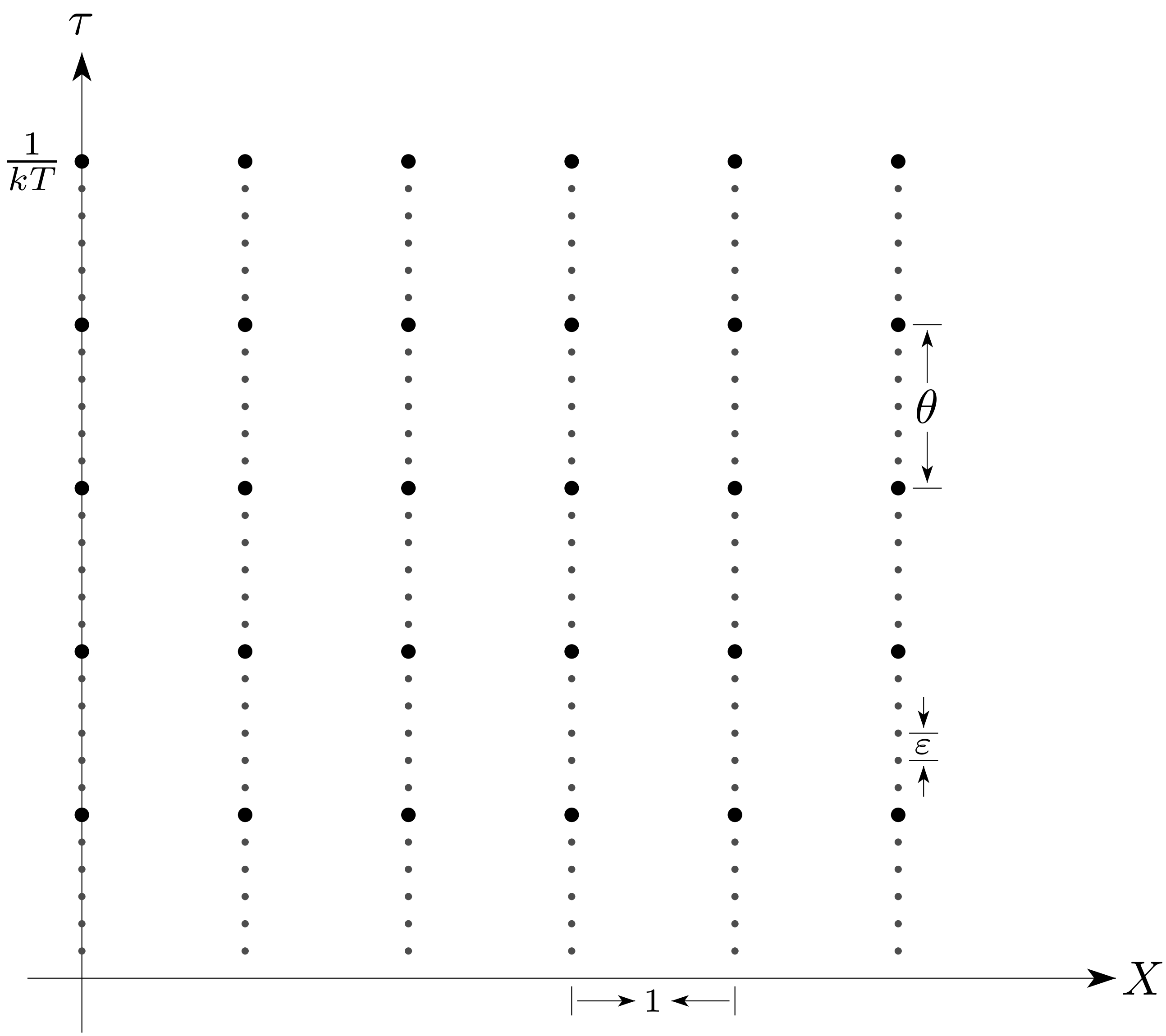
$$\chi(\psi^*,\psi) \textit{ cuts off} \quad \big|\psi_\tau(\mathbf{x})\big| \le \mathrm{R}(\varepsilon) \quad \big|\partial_0\psi_\tau(\mathbf{x})\big| \le \mathrm{r}(\varepsilon)$$

This step was completed more then half a dozen years ago.

The Construction Plan – The UV Regime

- *Temporal UV Limit:* Develop a picture of the temporal ultraviolet limit, $\varepsilon \to 0$ that you can work with. Evaluate enough of the integrals that, even after taking the limit $\varepsilon \to 0$, we have a more or less standard classical spin model on $\mathbb{Z}^{1+3}$. (Completed.) The nonstandard aspect of the model is that the spins, and the action, take values in $\mathbb{C}$, rather than $\mathbb{R}$.

- Final result of the Temporal UV Step is a representation of the partition function as an integral which involves ψ_τ only for $\tau \in \theta\mathbb{Z}$ for some θ independent of ε and which looks somewhat like a classical spin system.


τ
1/kT
θ
ε
X
1

We have shown (by repeated "decimation style" renormalization group steps)

- that the partition function can be written as

$$\mathrm{Tr}\; e^{-\frac{1}{kT}(H-\mu N)} = \int \prod_{\tau\in\theta\mathbb{Z}\cap(0,\frac{1}{kT}]} \Big[\prod_{\mathbf{x}\in X} \frac{d\psi_\tau(\mathbf{x})^*\wedge d\psi_\tau(\mathbf{x})}{2\pi\imath} e^{-\psi_\tau(\mathbf{x})^*\psi_\tau(\mathbf{x})} \Big] I_\theta(\psi^*_{\tau-\theta}, \psi_\tau)$$

- and that, if θ was chosen sufficiently small, we can write I_θ as the sum of a dominant part, $\mathcal{Z}_\theta^{|X|} e^{\mathcal{A}'_\theta}$ and terms indexed by proper subsets of X and which are nonperturbatively small, exponentially in the size of the subsets.
- For convenience, we rescale time so that the dominant ("small field") contribution is

$$\frac{1}{Z^{(n)}} \int \prod_{\substack{\tau\in\mathbb{Z}\cap(0,\frac{1}{\theta kT}] \\ \mathbf{x}\in X}} \frac{d\psi_\tau(\mathbf{x})^*\wedge d\psi_\tau(\mathbf{x})}{2\pi\imath}\; e^{\mathcal{A}_0(\psi^*,\psi)}\; \chi(\psi^*, \psi)$$

This step was completed about five years ago.

The Construction Plan – Crude Overview of the Thermodynamic Limit

Our plan is to take the infrared limit (i.e. $L_{\rm sp} \to \infty$) by

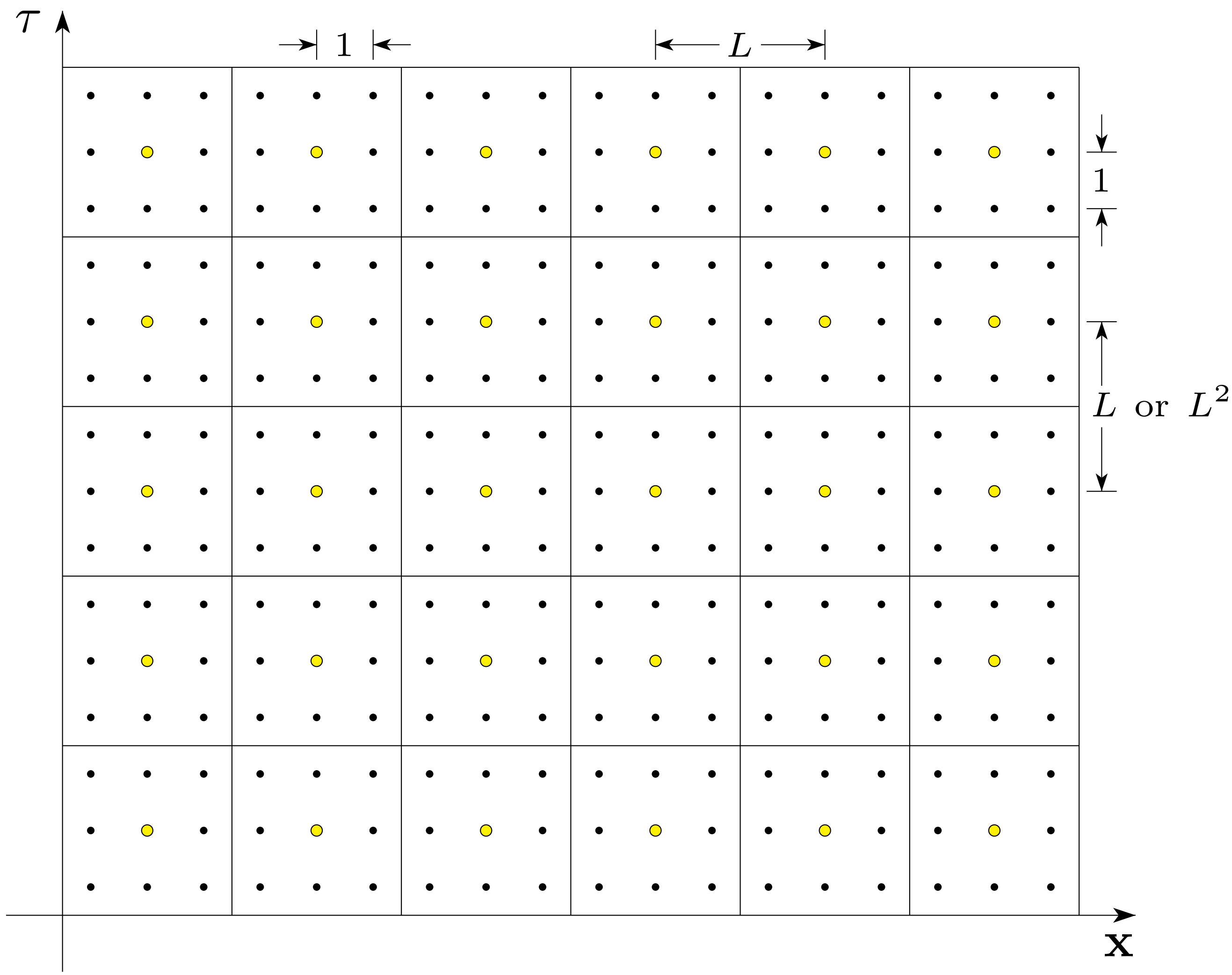
- repeatedly executing block spin renormalization group steps to successively "integrate out" lower and lower energy degrees of freedom.
- At the end of each renormalization group step we rescale so that the fields that have not yet been integrated out are indexed by a unit lattice.
- At the end of renormalization group step number n, the partition function is of the form

$$\mathrm{Tr}\ e^{-\frac{1}{kT}\,(H-\mu N)} = \frac{1}{Z^{(n)}}\int\Big[\prod_{x\in\mathcal{X}_0^{(n)}} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi i}\Big] F_n(\psi^*,\psi)$$

where

$$\begin{aligned}
\mathcal{X}_0^{(n)} &= \big(\mathbb{Z}/\tilde{\varepsilon}_n L_{\rm tp}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/\varepsilon_n L_{\rm sp}\mathbb{Z}^3\big)\\
\psi : \mathcal{X}_0^{(n)} &\to \mathbb{C}\\
F_n(\psi^*,\psi) &= \text{discussed shortly}\\
Z^{(n)} &= \text{a normalization constant}
\end{aligned}$$

A block spin renormalization group step



A block spin renormalization step consists of substituting

$$1 = \frac{1}{N_{\mathbb{T}}^{(n)}} \int \Big[\prod_{y \in \check{\mathcal{X}}^{(n)}} \frac{d\zeta(y)^* \wedge d\zeta(y)}{2\pi i} \Big] e^{-aL^{-2} \left\langle \zeta^* - Q^{(n)}\psi^*, \zeta - Q^{(n)}\psi \right\rangle_{\check{\mathcal{X}}^{(n)}}}$$

where

$$(Q^{(n)}\psi)(y) = \text{a (possibly smoothed) average of } \psi \text{ over a block of } \left\{ \begin{matrix} L^2 & \text{if } n < n_{\rm e} \\ L & \text{if } n \ge n_{\rm e} \end{matrix} \right\} \times L \times L \times L \text{ sites centred on } y$$

$$L = \text{a fixed odd integer}$$

$$n_{\rm e} = \left\{ \begin{matrix} \text{the scale at which the transition from} \\ \text{parabolic to elliptic scaling takes place} \end{matrix} \right.$$

$$N_{\mathbb{T}}^{(n)} = \text{a normalization constant}$$

$$\check{\mathcal{X}}^{(n)} = \left(\left\{ \begin{matrix} L^2 & \text{if } n < n_{\rm e} \\ L & \text{if } n \ge n_{\rm e} \end{matrix} \right\} \mathbb{Z}/\cdots \right) \times (L\mathbb{Z}^3/\cdots)$$

$$\zeta : \check{\mathcal{X}}^{(n)} \to \mathbb{C}$$

to give

$$\mathrm{Tr}\ e^{-\frac{1}{kT}\,(H-\mu N)} = \tfrac{1}{Z^{(n)}}\int \Big[\prod_{y\in\check{\mathcal{X}}^{(n)}} \tfrac{d\zeta(y)^*\wedge d\zeta(y)}{2\pi i}\Big](\mathbb{T}^{(n)}F_n)(\zeta^*,\zeta)$$

with

$$\begin{aligned}(\mathbb{T}^{(n)}&F_n)(\zeta_*,\zeta)\\ &= \tfrac{1}{N_{\mathbb{T}}^{(n)}}\int \Big[\prod_{x\in\mathcal{X}_0^{(n)}} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi i}\Big] e^{-aL^{-2}\left\langle \zeta_*-Q^{(n)}\psi^*,\zeta-Q^{(n)}\psi\right\rangle_{\check{\mathcal{X}}^{(n)}}} F_n(\psi^*,\psi)\end{aligned}$$

followed by the rescaling

$$\zeta\big(\tau,\mathbf{x}\big) = (\mathbb{S}_n^{-1}\tilde{\psi})\big(\tau,\mathbf{x}\big) = \begin{cases} L^{-\frac{3}{2}}\tilde{\psi}(L^{-2}\tau, L^{-1}\mathbf{x}) & \text{if } n < n_{\rm e} \\ L^{-1}\tilde{\psi}(L^{-1}\tau, L^{-1}\mathbf{x}) & \text{if } n \ge n_{\rm e} \end{cases}$$

All together, renaming $\tilde{\psi}\to\psi$,

$$\mathrm{Tr}\ e^{-\frac{1}{kT}\,(H-\mu N)} = \tfrac{1}{Z^{(n+1)}}\int \Big[\prod_{x\in\mathcal{X}_0^{(n+1)}} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi i}\Big](\mathbb{S}_n\mathbb{T}^{(n)}F_n)(\psi^*,\psi)$$

where $(\mathbb{S}_n\mathcal{F})(\psi_*,\psi) = \mathcal{F}\big(\mathbb{S}_n^{-1}\psi_*, \mathbb{S}_n^{-1}\psi\big)$.

Crude Outline of the Evaluation of $\mathbb{T}^{(n)} F_n$

- F_n is expected to be the sum of a dominant (pure small field) part, $F_n^{(\mathrm{SF})}$ and terms indexed by proper subsets of $\mathcal{X}_0^{(n)}$ and which are nonperturbatively small, exponentially in the size of the subsets. Consider just $F_n^{(\mathrm{SF})}$.
- $F_n^{(\mathrm{SF})} = e^{-\mathcal{A}_n + \mathcal{E}_n}$ where $\mathcal{A}_n$ is the dominant part of the action (details shortly) and $\mathcal{E}_n$ is an analytic function in the fields that is perturbatively small.
- Think of

$$\Big[\prod_{x\in\mathcal{X}_0^{(n)}} \int \tfrac{d\psi^*(x)\wedge d\psi(x)}{2\pi\imath}\Big] \quad \text{as} \quad \Big[\prod_{x\in\mathcal{X}_0^{(n)}} \int_{\psi_*(x)=\psi(x)^*} \tfrac{d\psi_*(x)\wedge d\psi(x)}{2\pi\imath}\Big]$$

To evaluate $\mathbb{T}^{(n)} F_n^{(\mathrm{SF})}$, which is

$$\tfrac{1}{N_{\mathbb{T}}^{(n)}}\Big[\prod_{x\in\mathcal{X}_0^{(n)}} \int_{\psi_*(x)=\psi(x)^*} \tfrac{d\psi_*(x)\wedge d\psi(x)}{2\pi\imath}\Big] e^{-aL^{-2}\langle \zeta_*-Q^{(n)}\psi_*,\zeta-Q^{(n)}\psi\rangle-\mathcal{A}_n+\mathcal{E}_n}$$

▷ Find the critical point of the map

$$(\psi_*, \psi) \mapsto aL^{-2} \langle \zeta_* - Q^{(n)}\psi_*, \zeta - Q^{(n)}\psi \rangle + \mathcal{A}_n(\psi_*, \psi)$$

[The existance, uniqueness and regularity properties that we need have been completely proven for the pure small field part of both the parabolic and elliptic flows.]

▷ We integrate only over a neighbourhood of the critical point. The remaining contributions are large field.

▷ Substitute

$$\psi_* = \psi_{*\mathrm{crit}}(\zeta_*, \zeta) + \delta\psi_* \qquad \psi = \psi_{\mathrm{crit}}(\zeta_*, \zeta) + \delta\psi$$

▷ Typically, $\psi_{*\rm crit}(\zeta_*,\zeta)$ is NOT the complex conjugate of $\psi_{\rm crit}(\zeta_*,\zeta)$ so that the domain of integration

$$\delta\psi_*(x) = \delta\psi(x)^* + \psi_{\rm crit}(\zeta_*,\zeta)^* - \psi_{*\rm crit}(\zeta_*,\zeta)$$

is not $\delta\psi_*(x) = \delta\psi(x)^*$. Apply Stokes' theorem to move the domain of integration to

$$\delta\psi_*(x) = \delta\psi(x)^*$$

[This step, including bounds on the boundary terms (which are included in the "large field" part of $F(\psi_*,\psi)$) has been rigorously executed in the UV regime. Preliminary bounds, but not the complete argument, have been done for the parabolic regime.]

▷ Do the integral. The result is $\frac{1}{\mathcal{Z}'_n} e^{-\check{\mathcal{A}}_{n+1}+\check{\mathcal{E}}_{n+1,1}+\tilde{\mathcal{E}}_{\rm fl}}$ where

$$\check{\mathcal{A}}_{n+1} = aL^{-2}\,\langle\zeta_* - Q^{(n)}\psi^*, \zeta - Q^{(n)}\psi\rangle_{\check{\mathcal{X}}^{(n)}} + \mathcal{A}_n(\psi_*,\psi)\big|_{\psi_{(*)\rm crit}}$$

$$\check{\mathcal{E}}_{n+1,1} = \mathcal{E}_n\big|_{\psi_{(*)\rm crit}}$$

$$\tilde{\mathcal{E}}_{\rm fl} = \text{"new" perturbatively sized contributions}$$

- ▷ Scale. That is, substitute $\zeta_{(*)} = \mathbb{S}_n^{-1}\psi_{(*)}$.
- ▷ In future steps, simply substituting the critical field and scaling will cause a small number of terms in $\tilde{\mathcal{E}}_{\mathrm{fl}}$ to grow. Move them into $\mathcal{A}_{n+1}$. That is renormalization. The parabolic regime is superrenormalizable:

$$\mu_n \approx \mu_0 L^{2n} \quad \mathrm{v}_n \approx \mathrm{v}_0 L^{-n}$$

The elliptic regime is strictly renormalizable. Naively,

$$\mu_n \approx \mu_0 L^{2n} \quad \mathrm{v}_n \approx \mathrm{v}_{n_\mathrm{e}}$$

[The detailed bounds that we need on functions like $\mathcal{E}_n$ and the result of the fluctuation integral

* have been completely proven for pure small field contributions in the parabolic regime
* have been roughly proven for pure small field contributions in the elliptic regime
* Renormalization has been completely treated for pure small field contributions in the parabolic regime.]

The Action

The dominant (pure small field) contribution to F_n is of the form

$$F_n^{(\mathrm{SF})}(\psi_*,\psi) = \exp\Big[-\mathcal{A}_n(\psi_*,\psi,\phi_{n*}(\psi^*,\psi),\phi_n(\psi^*,\psi)\big) + \mathcal{E}_n\big(\psi_*,\psi\big)\Big]$$

where in the parabolic regime

$$\begin{aligned}\mathcal{A}_n(\psi_*,\psi,\phi_*,\phi) = &\big\langle \psi_* - Q_n\phi_*, \mathfrak{Q}_n\big(\psi - Q_n\phi\big)\big\rangle_{\mathcal{X}_0} \\ &+\big\langle \phi_*, \big(-\partial_0 + \jmath_0(-\Delta)\big)\phi\big\rangle_{\mathcal{X}_n} + \cdots \\ &+ \tfrac{1}{2}\mathrm{v}_n\big\langle (\phi_*\phi)^2, 1\big\rangle_{\mathcal{X}_n} - \mu_n\,\langle(\phi_*\phi),1\rangle_{\mathcal{X}_n}\end{aligned}$$

and in the elliptic regime

$$\begin{aligned}\mathcal{A}_n(\psi_*,\psi,\phi_*,\phi) = &\big\langle \psi_* - Q_n\phi_*, \mathfrak{Q}_n\big(\psi - Q_n\phi\big)\big\rangle_{\mathcal{X}_0} \\ &+ \big\langle \phi_*, \big(-\delta_n^{-1}\partial_0 + \jmath_n(-\Delta)\big)\phi\big\rangle_{\mathcal{X}_n} \\ &+ \tfrac{1}{2}\mathrm{v}_n\big\langle (\phi_*\phi)^2, 1\big\rangle_{\mathcal{X}_n} - \mu_n\,\langle\phi_*\phi\,,\,1\rangle_{\mathcal{X}_n}\end{aligned}$$

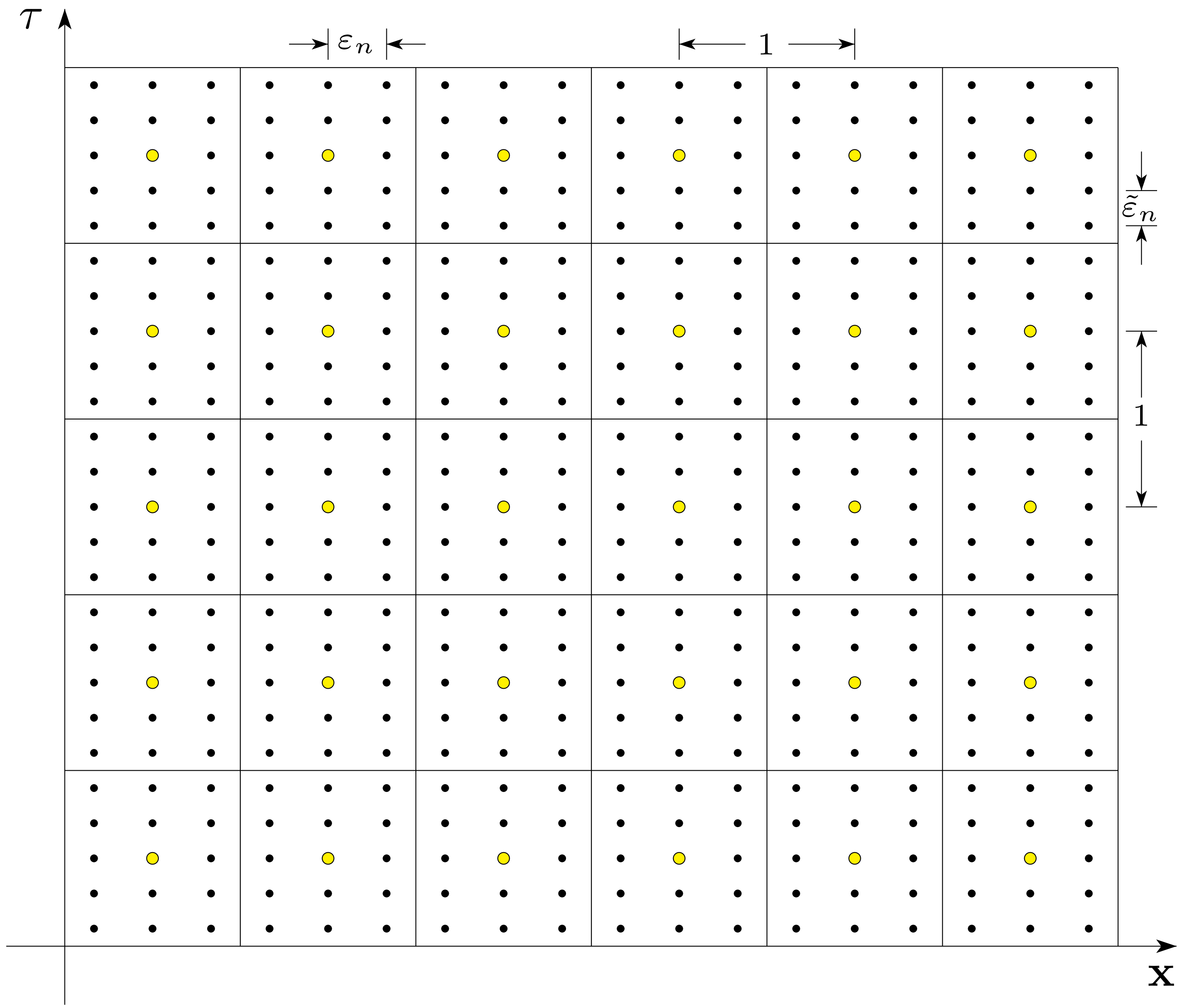
where

for $n \le n_\mathrm{e}$ $\quad \mu_n \approx \mu_0 L^{2n} \quad \mathrm{v}_n \approx \mathrm{v}_0 L^{-n} \quad \delta_n^{-1} = 1$

for $n \ge n_\mathrm{e}$ $\quad \mu_n \approx \mu_0 L^{2n} \quad \mathrm{v}_n \approx \mathrm{v}_{n_\mathrm{e}} \quad \delta_n^{-1} \approx L^{n-n_\mathrm{e}}$

and where

$$
\begin{aligned}
\mathcal{X}_0 &= \mathbb{Z} \times \mathbb{Z}^3 / \cdots \\
\mathcal{X}_n &= \text{the original lattice shrunk } n \text{ times by rescaling} \\
&= \big(\tilde{\varepsilon}_n \mathbb{Z}\big) \times \big(\varepsilon_n \mathbb{Z}^3\big) / \cdots \\
\varepsilon_n &= L^{-n} \\
\tilde{\varepsilon}_n &= \begin{cases} \varepsilon_n^2 = L^{-2n} & \text{if } n \le n_{\rm e} \\ \varepsilon_{n_{\rm p}} \varepsilon_n = L^{-n_{\rm e}-n} & \text{if } n \ge n_{\rm e} \end{cases} \\
n_{\rm e} &= \begin{cases} \text{the scale at which the transition from} \\ \text{parabolic to elliptic scaling takes place} \end{cases} \\
\langle \alpha, \beta \rangle_{\mathcal{X}_0} &= \sum_{x \in \mathcal{X}_0} \alpha(x) \beta(x) \\
\langle \alpha, \beta \rangle_{\mathcal{X}_n} &= \tilde{\varepsilon}_n \varepsilon_n^3 \sum_{u \in \mathcal{X}_n} \alpha(u) \beta(u) \\
Q_n &= \text{a block spin average which averages over blocks in } \mathcal{X}_n \\
&\qquad \text{that are centred on points of } \mathcal{X}_0 \\
\mathfrak{Q}_n &= \text{ an uninteresting operator } \approx \mathbb{1}
\end{aligned}
$$



τ
ε_n
1
$\tilde{\varepsilon}_n$
1
x

The Background Fields

The "background fields" $\phi_{n*}(\psi_*, \psi), \phi_n(\psi_*, \psi)$ are

- thought of as critical fields at one scale, evaluated at critical fields of the next scale, evaluated at critical fields of the next scale, $\cdots$
- obtained by solving

$$\frac{\partial}{\partial \phi_*} \mathcal{A}_n(\psi_*, \psi, \phi_*, \phi) = \frac{\partial}{\partial \phi} \mathcal{A}_n(\psi_*, \psi, \phi_*, \phi) = 0$$

This is a system of nonlinear "partial differential equations".

- In most of the parabolic regime the nonlinearity *can* be treated as a perturbation.
- In the last part of the parabolic regime and in the elliptic regime the nonlinearity *cannot be treated as a perturbation*. Instead we
  - switch to radial and tangential fields

$$\psi_* = r_n \exp\left\{\tfrac{1}{r_n}(\mathrm{R} - i\Theta)\right\} \qquad \psi = r_n \exp\left\{\tfrac{1}{r_n}(\mathrm{R} + i\Theta)\right\}$$

  where $r_n = \sqrt{\frac{\mu_n}{\mathrm{v}_n}}$ is the radius of the potential well.

- ▷ Make an initial guess Φ_*, Φ for ϕ_*, ϕ. They are determined from ψ_*, ψ by a carefully chosen system of linear "partial differential equations".
- ▷ Perturb off of Φ_*, Φ, again using radial and tangential fields.

$$\phi_* = \Phi_* \exp\left\{\tfrac{1}{r_n}(\delta_n\chi - i\eta)\right\} \qquad \phi = \Phi \exp\left\{\tfrac{1}{r_n}(\delta_n\chi + i\eta)\right\}$$

[The existance, uniqueness and regularity properties that we need have been completely proven for the pure small field part of both the parabolic and elliptic flows.]

The Elliptic Background Field Equations

Write

$$\begin{aligned} \psi_* &= r_n \exp\left\{\tfrac{1}{r_n}(\mathrm{R} - i\Theta)\right\} & \psi &= r_n \exp\left\{\tfrac{1}{r_n}(\mathrm{R} + i\Theta)\right\} \\ \Phi_* &= r_n \exp\left\{\tfrac{1}{r_n}(\delta_n \mathrm{X} - i\mathrm{H})\right\} & \Phi &= r_n \exp\left\{\tfrac{1}{r_n}(\delta_n \mathrm{X} + i\mathrm{H})\right\} \\ \phi_* &= \Phi_* \exp\left\{\tfrac{1}{r_n}(\delta_n \chi - i\eta)\right\} & \phi &= \Phi \exp\left\{\tfrac{1}{r_n}(\delta_n \chi + i\eta)\right\} \end{aligned}$$

Then the background field equations, in the elliptic regime, are

$$\Box_n \begin{bmatrix} \chi \\ \eta \end{bmatrix} + \begin{bmatrix} \delta_n \mathrm{V}_1 \\ \mathrm{V}_2 \end{bmatrix} = \begin{bmatrix} \delta_n \mathbb{F}_1 \\ \mathbb{F}_2 \end{bmatrix} + Q_n^\star \mathfrak{Q}_n \begin{bmatrix} \delta_n \mathrm{R} \\ \Theta \end{bmatrix} - \Box_n \begin{bmatrix} \mathrm{X} \\ \mathrm{H} \end{bmatrix}$$

which reduces to

$$\Box_n \begin{bmatrix} \chi \\ \eta \end{bmatrix} + \begin{bmatrix} \delta_n \mathrm{V}_1 \\ \mathrm{V}_2 \end{bmatrix} = \begin{bmatrix} \delta_n \mathbb{F}_1 \\ \mathbb{F}_2 \end{bmatrix}$$

when we choose $\Phi_{(*)}$ by

$$\Box_n \begin{bmatrix} \mathrm{X} \\ \mathrm{H} \end{bmatrix} = Q_n^\star \mathfrak{Q}_n \begin{bmatrix} \delta \mathrm{R} \\ \Theta \end{bmatrix}$$

Here $\mathrm{V} = \begin{bmatrix} \mathrm{V}_1 & \mathrm{V}_2 \end{bmatrix}^t$ and $\mathbb{F} = \begin{bmatrix} \mathbb{F}_1 & \mathbb{F}_2 \end{bmatrix}^t$ are explicitly constructed (but complicated) entire functions and

$$\square_n = \begin{bmatrix} 2\delta_n^2\mu_n + \frac{\tilde{\varepsilon}_n\delta_n}{2}\partial_0^\star\partial_0 + \jmath_n\delta_n^2(-\Delta) & -\frac{i}{2}\big(\partial_0 - \partial_0^\star\big) \\ \frac{i}{2}\big(\partial_0 - \partial_0^\star\big) & \frac{\tilde{\varepsilon}_n}{2\delta_n}\partial_0^\star\partial_0 + \jmath(-\Delta) \end{bmatrix} + Q_n^\star \mathfrak{Q}_n Q_n \begin{bmatrix} \delta_n^2 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\Theta : \mathbb{Z} \times \mathbb{Z}^3 \to \mathbb{C}$$

$$\mathrm{H} : \tilde{\varepsilon}_n\mathbb{Z} \times \big(\varepsilon_n\mathbb{Z}\big)^3 \to \mathbb{C}$$

$$R : \big(\mathbb{Z}/\tilde{\varepsilon}_n L_{\mathrm{tp}}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/\varepsilon_n L_{\mathrm{sp}}\mathbb{Z}^3\big) \to \mathbb{C}$$

$$\mathrm{X}\,,\ \chi\,,\ \eta : \tilde{\varepsilon}_n\mathbb{Z}/\tilde{\varepsilon}_n L_{\mathrm{tp}}\mathbb{Z} \times \big(\varepsilon_n\mathbb{Z}/\varepsilon_n L_{\mathrm{sp}}\mathbb{Z}\big)^3 \to \mathbb{C}$$

$$e^{i\Theta/r_n}\,,\ \partial_\nu\Theta : \big(\mathbb{Z}/\tilde{\varepsilon}_n L_{\mathrm{tp}}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/\varepsilon_n L_{\mathrm{sp}}\mathbb{Z}^3\big) \to \mathbb{C}$$

$$e^{i\mathrm{H}/r_n}\,,\ \partial_\nu\mathrm{H}\,,\ \mathrm{H}_\Theta : \tilde{\varepsilon}_n\mathbb{Z}/\tilde{\varepsilon}_n L_{\mathrm{tp}}\mathbb{Z} \times \big(\varepsilon_n\mathbb{Z}/\varepsilon_n L_{\mathrm{sp}}\mathbb{Z}\big)^3 \to \mathbb{C}$$

where

$$\mathrm{H}_\Theta(u) = \mathrm{H}(u) - \Theta\big(x(u)\big)$$