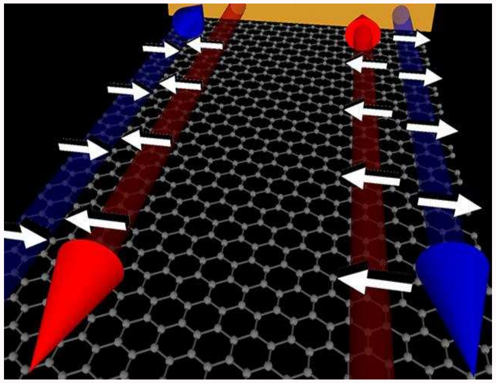
Topological invariants of Floquet systems

Krzysztof Gawędzki, IHP, November 2015

dedicated to Vincent Rivasseau for the 60th anniversary

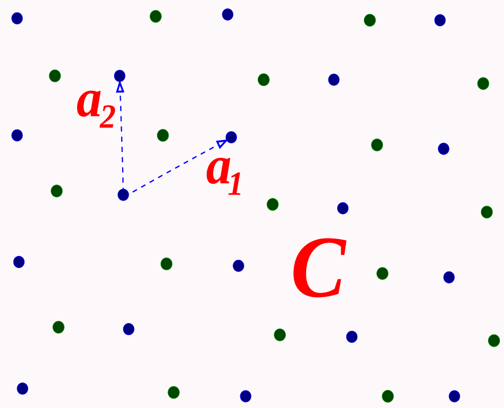
- ## Topological insulators

  Materials with topological order that are insulators in their interior but have edges carrying protected conducting states



- ## Simplest setup

  1^{st} quantized models on infinite crystalline lattice $\mathcal{C} \subset \mathbf{E}^d$



- ## Bloch theory

  - Diagonalization of discret translations by **Fourier** transform gives the space of states

$$L^2(\mathbb{T}^d, \mathbf{C}^N)$$

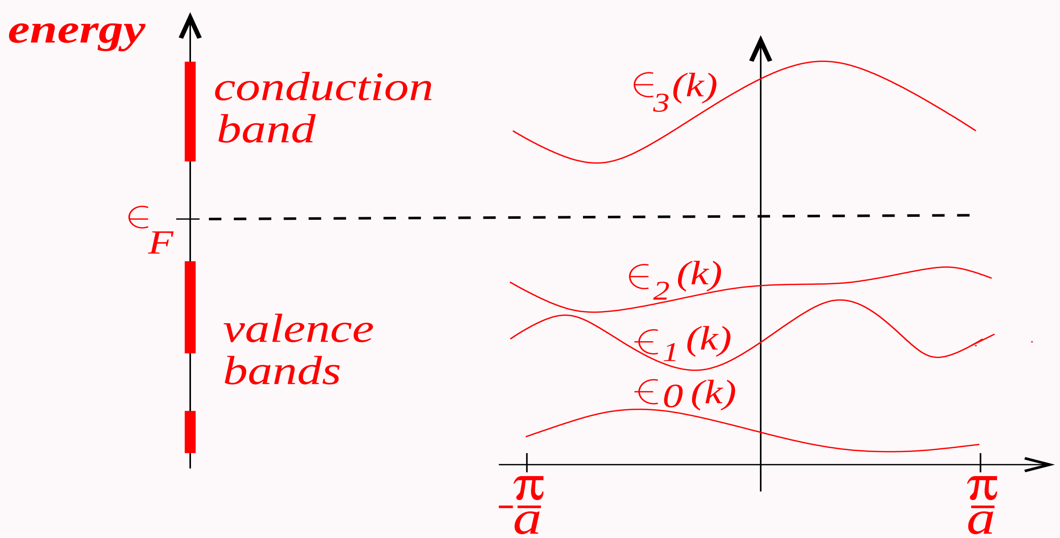
where $\mathbb{T}^d$ is the **Brillouin** torus of quasi-momenta $k \bmod \frac{2\pi}{a}\mathbb{Z}^d$

- The evolution is governed by lattice **Hamiltonian** H that in the **Bloch** picture is block-diagonal

$$(H\psi)(k) \,=\, H(k)\,\psi(k)$$

for $\psi \in L^2(\mathbb{T}^d, \mathbb{C}^N)$ where $H(k)$ are $N \times N$ hermitian matrices smoothly dependent on $k \in \mathbb{T}^d$

- **Insulators** have a gap in the spectrum of $H(k)$ around the **Fermi** energy ϵ_F separating eigenvalues $e_n(k) < \epsilon_F$ from $e_n(k) > \epsilon_F$



- **Chern topological insulators**

  - Spectral projectors $P(k)$ on $e_n(k) < \epsilon_F$ smoothly depend on $k \in \mathbb{T}^d$

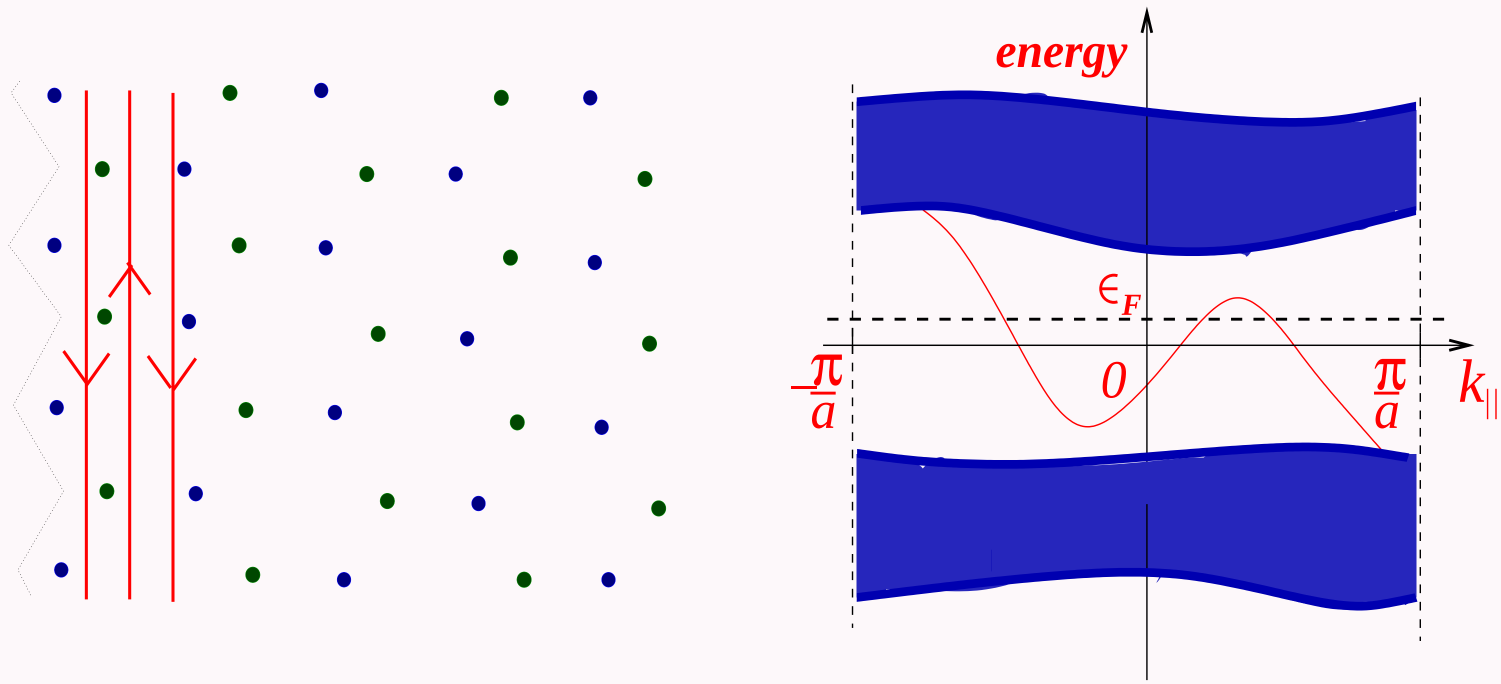
  - Subspaces $\mathcal{E}(k) = P(k)\,\mathbb{C}^N \subset \mathbb{C}^N$ form a **vector bundle** $\mathcal{E}$

  - The simplest topological invariant for $2d$ insulators (like doped **graphene**) is the 1^{st} **Chern number** of $\mathcal{E}$

$$c_1(\mathcal{E}) \;=\; \frac{\mathrm{i}}{2\pi}\int_{\mathbb{T}^2} \mathbf{tr}\, P(k)\,\big(dP(k)\big)^{\wedge 2} \;\;\in\;\; \mathbb{Z}$$

  - $\frac{\mathrm{i}}{2\pi}\,\mathbf{tr}\,P(dP)^2$ $\propto$ to the **Berry** curvature is the simplest example of a **characteristic class**

- **Bulk-edge correspondence**: $c_1(\mathcal{E})$ counts with chirality the massless states at the boundary components of a bounded crystal with energies in the bulk gap



- **Floquet generalization of Chern insulators**

  - Perturbation periodic in time (e.g. due to a microwave) lead to time-dependent Hamiltonians $H(t) = H(t+T)$

  - The evolution operators $\mathrm{i}\partial_t U(t) = H(t)\,U(t)$, $U(0) = I$, satisfy $U(t+T) = U(t)\,U(T)$

  - **Floquet theory:** spectral analysis of $U(T,k)$ with eigenvalues $\mathrm{e}^{-\mathrm{i}\,e_n(k)T}$ replaces that of static **Bloch** Hamiltonians $H(k)$

  - **Quasi-energies** $e_n(k)$ are defined modulo $\frac{2\pi}{T}$ $\Rightarrow$ quasi-energy bands repeat themselves periodically

- **gapped** **Floquet** **system**: $U(T,k)$ have a spectral gap around quasi-energy ϵ for all k

- Gap-dependent static **effective Hamiltonians**

$$H_\epsilon(k) \;=\; \tfrac{i}{T}\,\ln_{-\epsilon T} U(T,k)$$

cut at argument $-\epsilon T$

satisfy $U(T,k) = \mathrm{e}^{-\mathrm{i}T\,H_\epsilon(k)}$ and are used to periodize the evolution:

$$V_\epsilon(t,k) \;=\; U(t,k)\;\mathrm{e}^{it\,H_\epsilon(k)} \;=\; V_\epsilon(t+T,k)$$

- Let $\chi = \frac{1}{12\pi}\,\mathbf{tr}(g^{-1}dg)^{\wedge 3}$ be a closed 3-form on $U(N)$
Rudner-**Lindner**-**Berg**-**Levin** (2013) considered the "degree"

$$W_\epsilon \;=\; \frac{1}{2\pi}\int_{\mathbb{T}^3} V_\epsilon^*\chi \;\in\; \mathbb{Z}$$

of V_ϵ as a dynamical **invariant** of $2d$ **Floquet** systems

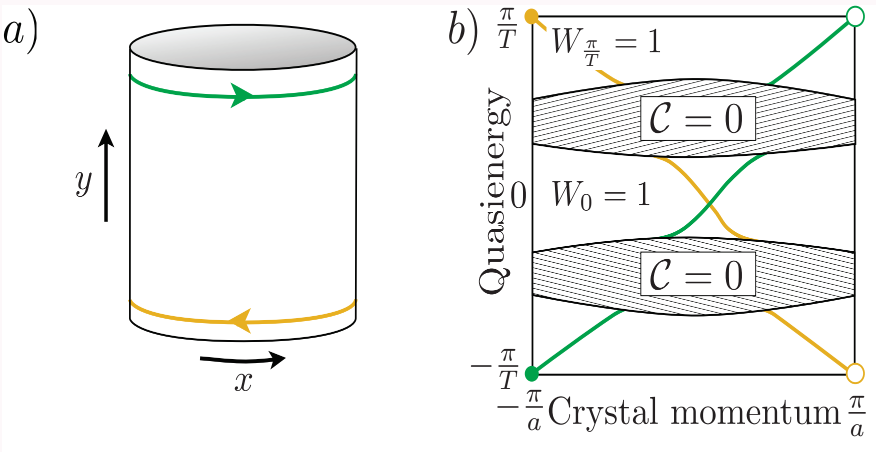
- For $0 < \epsilon < \epsilon' < \frac{2\pi}{T}$

$$H_{\epsilon'}(k) - H_{\epsilon}(k) = \frac{2\pi}{T} P_{\epsilon,\epsilon'}(k)$$

$P_{\epsilon,\epsilon'}(k)$ - spectral proj. of $U(T,k)$ on quasi-energies $\epsilon < e_n(k) < \epsilon'$

- Implies that $W_{\epsilon'} - W_{\epsilon} = c_1(\mathcal{E}_{\epsilon,\epsilon'})$

- W_ϵ counts with chirality the **edge states** of $U(T)$ in the bulk quasi-energy gap around ϵ



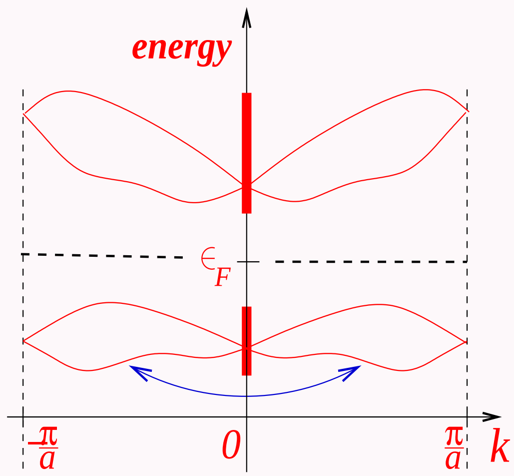
• **Time-reversal invariant topological insulators**

- Time reversal sends $\psi(k)$ to $\theta\psi(-k)$ where θ is an anti-unitary operator in $\mathbf{C}^N$ s. t. $\theta^2 = -I$ (N must be even)

- For time-reversal invariant (TRI) systems

$$\theta\, H(k)\, \theta^{-1} \;=\; H(-k)$$

- Eigenvectors of $H(k)$ and of $H(-k)$ come then in **Kramers' pairs** $\psi_n(k)$ and $\theta\psi_n(k)$ with the same energy



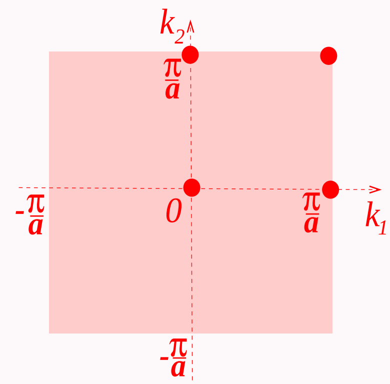
- TRI implies that $\theta P(k)\theta^{-1} = P(-k) \quad \Rightarrow \quad c_1(\mathcal{E}) = 0$
  $\Rightarrow \quad \mathcal{E}$ is trivializable so that $\exists$ a global frame

$$(\psi_i(k)), \;\; i = 1, \dots, 2m$$

- **Kane**-**Mele** (2005) defined an invariant $\mathrm{KM}(\mathcal{E}) \in \boldsymbol{Z}_2$ obstructing the choice of a global frame of $\mathcal{E}$ composed of **Kramers**' pairs

- Let $w_{ij}(k) = \langle \psi_i(-k) | \theta \psi_j(k) \rangle = -w_{ji}(-k)$ ("**sewing matrix**")
  Obstruction $\mathrm{KM}(\mathcal{E})$ was given by the expression (**Fu**-**Kane** 2006)

$$(-1)^{\mathrm{KM}(\mathcal{E})} = \prod_{\substack{4 \ \mathrm{TRIM} \\ k=-k \in \mathbb{T}^2}} \frac{\sqrt{\det(w(k))}}{\mathrm{pf}(w(k))}$$



- $\mathrm{KM}(\mathcal{E})$ counts modulo 2 the number of **Kramers**' pairs of massless **edge states** of opposite chirality (**Graf**-**Porta** 2013)

- **Questions**

  - Are there characteristic-class-type integral formulae for $\mathrm{KM}(\mathcal{E})$ **?**

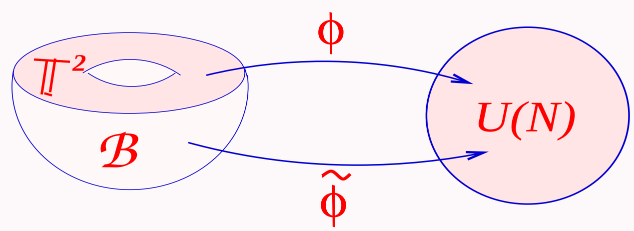
  - How to generalise $\mathrm{KM}(\mathcal{E})$ to TRI **Floquet** systems **?**

- **Interlude I.** **Wess-Zumino amplitudes and their square root**

  - $\forall\ \ \phi : \mathbb{T}^2 \to U(N)$. $\exists$ its extension $\widetilde{\phi} : \mathcal{B} \to U(N)$ to an oriented 3-manifold $\mathcal{B}$ with $\partial\mathcal{B} = \mathbb{T}^2$ such that

$$\exp\left[\mathrm{i}\int_{\mathcal{B}} \widetilde{\phi}^{\,*}\chi\right] \;\equiv\; \mathrm{e}^{iS_{\mathrm{WZ}}(\phi)}$$

    is independent of the choice of the extension and defines the $2d$ **Wess-Zumino** amplitude of ϕ (**Witten** 1983)



- Suppose that ϕ is TRI

$$\phi(-k) \;=\; \theta\, \phi(k)\, \theta^{-1}$$

$\exists\;\; \mathcal{B}$ with $\partial\mathcal{B} = \mathbb{T}^2$ with an orientation preserving involution ϑ reducing to $k \mapsto -k$ on $\partial\mathcal{B}$ and an extension $\widetilde{\phi} : \mathcal{B} \to U(N)$ of ϕ s. t.

$$\widetilde{\phi}(\vartheta(x)) \;=\; \theta\, \widetilde{\phi}(x)\, \theta^{-1}$$

and

$$\exp\left[\tfrac{\mathrm{i}}{2} \int_{\mathcal{B}} \widetilde{\phi}^{\,*} \chi\right] \;\equiv\; \sqrt{\mathrm{e}^{iS_{\mathrm{WZ}}(\phi)}}$$

is independent of the choice of $\widetilde{\phi}$ and defines **the** square root of the **Wess-Zumino** amplitude of ϕ

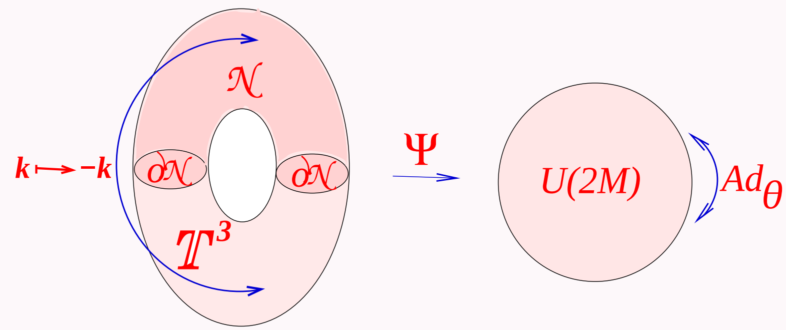
Proof. By direct construction and checking

• Interlude II. $3d$ index

• Let $\Psi:\mathbb{T}^3\to U(N)$ be such that

$$\Psi(-k)\;=\;\theta\,\Psi(k)\,\theta^{-1}$$

Let $\mathcal{N}\subset\mathbb{T}^3$ be a half of $\mathbb{T}^3$ forming a fundamental domain of $k\mapsto -k$



Proposition.

$$\mathcal{K}(\Psi)\;\equiv\;\frac{\exp\left[\frac{i}{2}\int_{\mathcal{N}}\Psi^*\chi\right]}{\sqrt{\mathrm{e}^{iS_{\mathrm{WZ}}(\Psi|_{\partial\mathcal{N}})}}}\;\in\;\{\pm 1\}$$

does not depend on choice of $\mathcal{N}$

Proof. By gerbe techniques using local expressions for $\sqrt{\mathrm{e}^{iS_{\mathrm{WZ}}}}$

- **Integral formulae for** $\mathrm{KM}(\mathcal{E})$

  - One has the relation (apparently unknown)

$$(-1)^{\mathrm{KM}(\mathcal{E})} \;=\; \mathrm{e}^{iS_{\mathrm{WZ}}(\underset{\textbf{sewing matrix}}{w})}$$

  - Let for a family of the valence band projectors $P(k)$

$$\Psi_P(t,k) \;=\; \mathrm{e}^{\mathrm{i}tP(k)} \;=\; \Psi_P(t+2\pi,k)$$

$$\phi_P(k) = \Psi_P(\pi,k) = I - 2P(k)$$

    If $P(-k) = \theta\, P(k)\, \theta^{-1}$ then

$$(-1)^{\mathrm{KM}(\mathcal{E})} \;=\; \sqrt{\mathrm{e}^{\mathrm{i}S_{\mathrm{WZ}}(\phi_P)}} \;=\; \mathcal{K}(\Psi_P)$$

• **Floquet generalization of** $\mathrm{KM}(\mathcal{E})$

joint work with **D. Carpentier**, **P. Delplace**, **M. Fruchart** and **C. Tauber**,
PRL **114** (2015), 106806 and Nucl. Phys. B **896** (2015), 779-834

- For TRI periodic systems with $H(t) = H(t+T)$ s. t. $H(-t) = \theta H(t)\theta^{-1}$ the periodized evolution satisfies

$$V_\epsilon(-t,-k) \;=\; \theta\, V_\epsilon(t,k)\, \theta^{-1}$$

$\Rightarrow$ in $2d$ the **RLBL** index $W_\epsilon = 0$

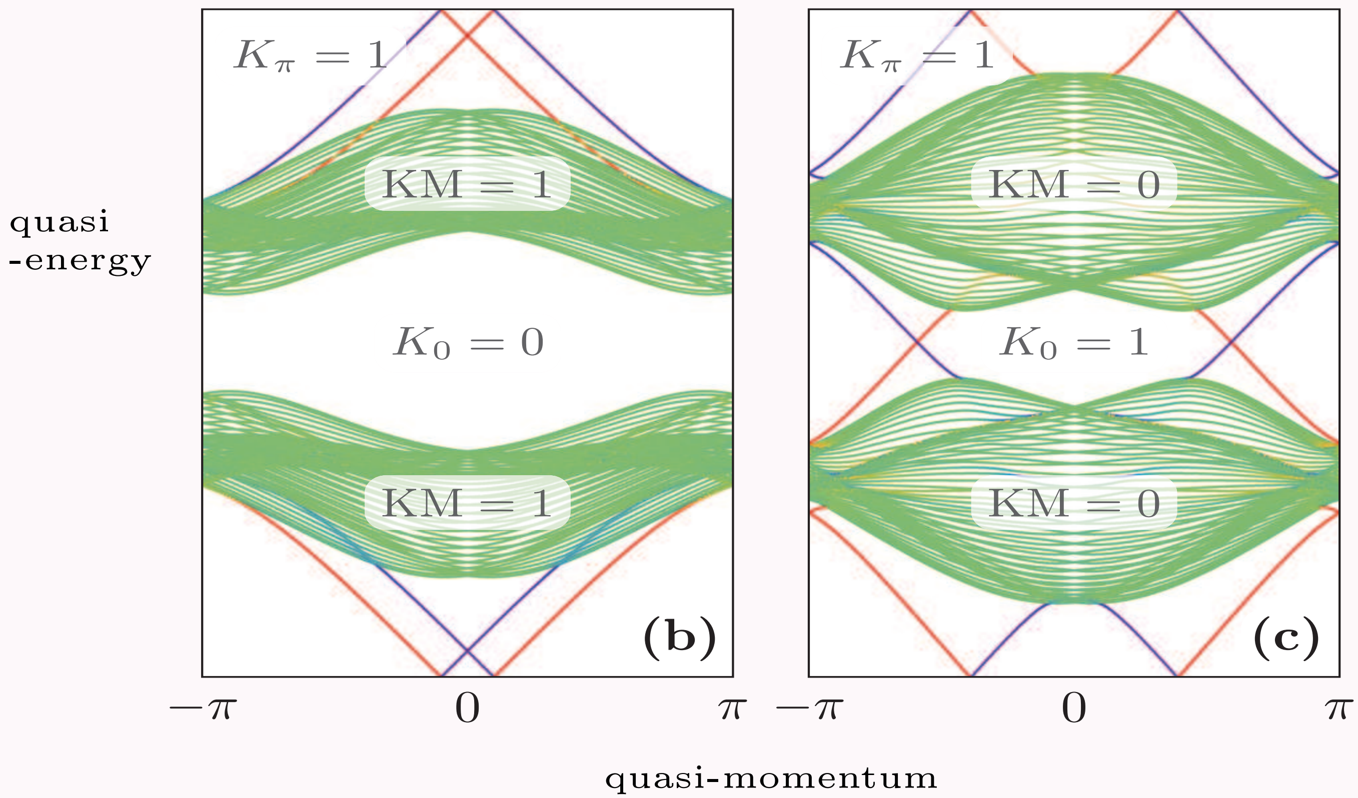
- We defined in this case the dynamical topological index $K_\epsilon \in \mathbb{Z}_2$ by

$$(-1)^{K_\epsilon} \;=\; \mathcal{K}(V_\epsilon)$$

- The **RLBL** relation $W_{\epsilon'} - W_\epsilon = c_1(\mathcal{E}_{\epsilon,\epsilon'})$ is replaced now by

$$K_{\epsilon'} - K_\epsilon \;=\; \mathrm{KM}(\mathcal{E}_{\epsilon,\epsilon'})$$

- K_ϵ should count modulo 2 the **Kramers**'s **pairs** of edge states of $U(T)$ inside the bulk quasi-energy gap labeled by ϵ



- **Extension to $3d$**

  - Strong **Fu-Kane-Mele** invariant (2007) $\mathrm{KM}^s(\mathcal{E})$ is given by the same formula as in $2d$ but with 8 TRIM

  - Relation to **Chern-Simons** amplitudes (magnetoelectric polarizability)

$$(-1)^{\mathrm{KM}^s(\mathcal{E})} \;=\; \exp\left[\mathrm{i}\int_{\mathbb{T}^3} S_{\mathrm{CS}}(A^B)\right]$$

Berry connection

  - New integral expression using our $3d$ index:

$$(-1)^{\mathrm{KM}^s(\mathcal{E})} \;=\; \mathcal{K}(\phi_P)$$

  - Strong invariant for $3d$ **Floquet** systems:

$$(-1)^{K^s_\epsilon} \;=\; \mathcal{K}(V_\epsilon|_{t=\frac{T}{2}}), \qquad K^s_{\epsilon'} - K^s_\epsilon \;=\; \mathrm{KM}^s(\mathcal{E}_{\epsilon,\epsilon'})$$

  - Main remaining open problem: extending the relations to **WZ** amplitudes to the bulk-edge correspondence

Many happy returns, Vincent