A solvable QFT in 4 D

How much Vincent influenced our work...

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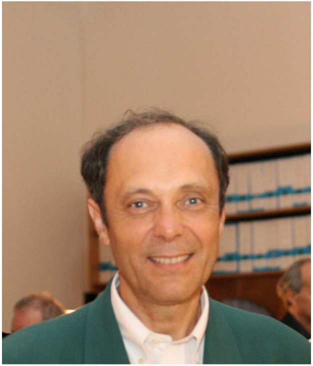
(based on joint work with Raimar Wulkenhaar,
arXiv: 1205.0465, 1306.2816, 1402.1041, 1406.7755 & 1505.05161)

to VINCENT

Congratulation to your

60 th birthday on 5 th of Dec



- Studies at ENS 74-78
- Studies at Princeton 78-79
- with A Wightman...Triviality?

Introduction

Prove that a non-trivial toy model for a quantum field theory on $\mathbb{R}^4$ exists and satisfies [O-S, Wightman].

- Φ^4_4 is renormalizable, $\Phi^4_{4+\epsilon}$ trivial
- Φ^4_4 on Moyal space is nonrenormalizable due to IR/UV mixing
- $(\Phi^4_4)_{\mathit{modified}}$ on Moyal space is renormalizable

  H G + R Wulkenhaar, 2004
- β function is perturbative zero (at $\Omega = 1$)

  M Disertori, R Gurau, J Magnen, V Rivasseau, 2006
- Ward identities allow to decouple SD equs.

$\rightarrow$ Can we construct it?

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- Ward identities allow to decouple SD equs.

$\rightarrow$ Can we construct it? ... YES

To our big surprise, Wightman axioms seem to be satisfied!

History

summer 2004: Vincent visits MPI Leipzig, met R Wulkenhaar

- 0409312 Non-Commutative Renormalization V. Rivasseau, F. Vignes-Tourneret
- 0501036 Renormalization of noncommutative ϕ^4-theory by multi-scale analysis V. Rivasseau, F. Vignes-Tourneret, R. Wulkenhaar
- 0512071 Propagators for Noncommutative Field Theories R. Gurau, V. Rivasseau, F. Vignes-Tourneret
- 0512271 Renormalization of Non-Commutative Phi^4_4 Field Theory in x Space R Gurau, J Magnen, V Rivasseau, F Vignes-Tourneret
  derived Symanzik polynomials Gurau:2006,
- Gross-Neveu model VignesTourneret:2006
- 0610224 Two and Three Loops Beta Function of Non Commutative ϕ^4_4 Theory M Disertori, V Rivasseau

0612251 Vanishing of Beta Function of Non Commutative ϕ^4_4 Theory to all orders M. Disertori, R. Gurau, J. Magnen, V. Rivasseau

0701034 Parametric representation of "critical" noncommutative QFT models V Rivasseau, A Tanasa
0702068 Renormalisation of non-commutative field theories V Rivasseau, F Vignes-Tourneret
0705.0705 Non-commutative Renormalization Vincent Rivasseau
0705.3437 Non-Commutative Complete Mellin Representation for Feynman Amplitudes R. Gurau, A.P.C. Malbouisson, V. Rivasseau, A. Tanasa
0711.1748 Why Renormalizable NonCommutative Quantum Field Theories? V Rivasseau
0807.4093 Commutative limit of a renormalizable noncommutative model J Magnen, V Rivasseau, A Tanasa
0806.4255 Non Commutative Field Theory on Rank One Symmetric Spaces P. Bieliavsky, R. Gurau, V. Rivasseau
0805.4362 Vanishing beta function for Grosse-Wulkenhaar model in a magnetic field J Ben Geloun, R Gurau, V Rivasseau
0805.2538 Color Grosse-Wulkenhaar models: One-loop β-functions J Ben Geloun, V Rivasseau
0802.0791 A translation-invariant renormalizable non-commutative scalar model R. Gurau, J. Magnen, V. Rivasseau, A. Tanasa
1104.3443 Constructive Renormalization for Φ^4_2 Theory with Loop Vertex Expansion Vincent Rivasseau, Zhituo Wang

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4} dx \Big(\frac{1}{2}\phi\ \big(-\Delta \qquad\qquad + \mu^2 \quad\big)\phi + \frac{\lambda}{4}\phi\ \ \phi\ \ \phi\ \ \phi\Big)(x)$$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi] = \int_{\mathbb{R}^4} dx \Big(\frac{1}{2}\phi \ \ (-\Delta + \Omega^2(\quad x)^2 + \mu^2 \quad)\phi + \frac{\lambda}{4}\phi \ \ \phi \ \ \phi \ \ \phi \Big)(x)$$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4} dx\Big(\frac{1}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2\quad)\phi+\frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

Regularisation of $\lambda\phi^4_4$ on noncommutative space

$$S[\phi]=\int_{\mathbb{R}^4} dx \Big(\frac{Z_\Lambda}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z_\Lambda^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

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matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\, f_{\underline{mn}}(x)=V\delta_{\underline{mn}}$

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takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

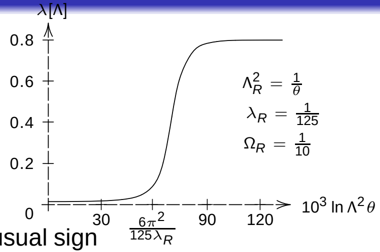
due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\, f_{\underline{mn}}(x)=V\delta_{\underline{mn}}$ the form

$$S[\Phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}H_{\underline{m}}\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z_\Lambda^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

$$H_{\underline{m}}=Z_\Lambda\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big)\,,\qquad |\underline{m}|:=m_1+m_2\leq\mathcal{N}$$

- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the nc manifold.

The β-function



one-loop calculation

$$\frac{d\lambda}{d\Lambda} = \beta_\lambda = \lambda^2 \frac{(1-\Omega^2)}{(1+\Omega^2)^3} + \mathcal{O}(\lambda^3)$$

$\lambda[\Lambda]$ diverges in commutative case

four-point function renormalisation with usual sign

$\exists$ one-loop wavefunction renormalisation which compensates four-point function renormalisation for $\Omega \to 1$

Euclidean quantum field theory

for unbounded positive selfadjoint operator H with compact resolvent

partition function $\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))$

Perturbative expansion $e^{-V\,\mathrm{tr}(P[\Phi])} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(V\,\mathrm{tr}(P[\Phi]))^n$

leads to ribbon graphs. They encode genus-g Riemann surface with B boundary components

We avoid the expansion, but keep the topological structure!

Ward identity

Matrix Base $\quad H_{nm} = Z_\Lambda\left(\frac{(n+m)}{V} + \frac{\mu^2_{bare}}{2}\right)$

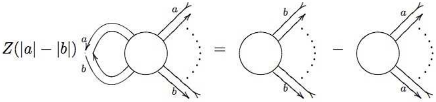
$$S[\Phi] = \sum_{n,m} \frac{1}{2}\Phi_{nm}H_{nm}\Phi_{mn} + \frac{\lambda}{4}\sum_{nmpq} Z^2_\Lambda \Phi_{nm}\Phi_{mp}\Phi_{pq}\Phi_{qn}$$

inner automorphism $\phi \mapsto U^*\Phi U$ of M_Λ, infinitesimally,
not a symmetry of the action, but invariance of measure

Interpretation

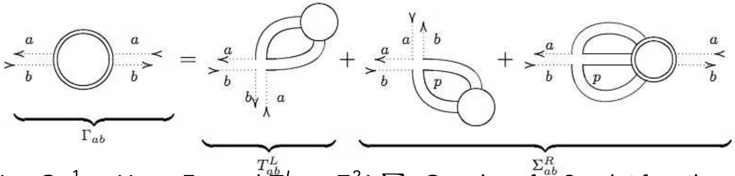
Insertion of special vertex $V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$

into external face equals the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$



The dots stand for the remaining face indices.

SD equation 2



Use $G^{-1}_{ab} = H_{ab} - \Gamma_{ab}$ and $T^L_{ab} = Z^2_\Lambda \lambda \sum_q G_{aq}$ gives for 2 point function:

$$Z^2_\Lambda \lambda \sum_q G_{aq} - Z_\Lambda \lambda \sum_p (G_{ab})^{-1} \frac{G_{bp} - G_{ba}}{|p| - |a|} = H_{ab} - G^{-1}_{ab} \,.$$

express SD equation in terms of the 1PI function Γ_{ab}, perform nonperturbative renormalisation ! for the 1PI part, Taylor expand Γ_{ab},

1PI four-point function

$$\Gamma^{ren}_{abcd} = Z_\Lambda \lambda \Big\{ \frac{G^{-1}_{ad} - G^{-1}_{cd}}{|a| - |c|} + \sum_p \frac{G_{pb}}{|a| - |p|} \Big(\frac{G_{dp}}{G_{ad}} \Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd} \Big) \Big\}$$

Take limits,...

Schwinger-Dyson equations (for $S_{int}[\Phi] = \frac{\lambda}{4}\mathrm{tr}(\Phi^4)$)

In a scaling limit $V \to \infty$ and $\frac{1}{\sqrt{V}} \sum_{p \in I}$ finite, we have:

1. A closed non-linear equation for $G_{|ab|}$

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p \in I} \Big(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \Big)$$

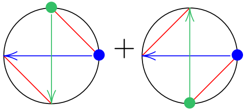
2. For $N \geq 4$ a universal algebraic recursion formula

$$G_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

- scaling limit corresponds to restriction to genus $g = 0$
- similar formulae for $B \geq 2$
- no index summation in $G_{|abcd|} \quad \Rightarrow \quad \beta$-function zero!

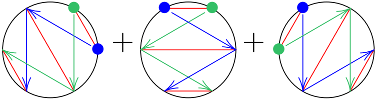
Graphical realisation

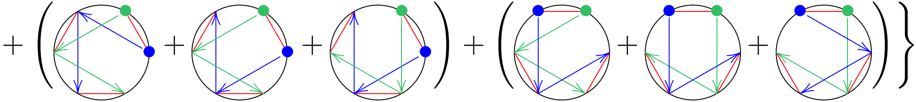
$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3} - G_{b_0b_3}G_{b_2b_1}}{(b_0-b_2)(b_1-b_3)} = -\lambda\Big\{$$



$$\Big\}$$

$$G_{b_0\dots b_5} = \lambda^2\Big\{$$





$$\Big\}$$

$$= G_{b_ib_j}$$

leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

$$= \frac{1}{b_i - b_j}$$

leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Back to $\lambda\Phi^4_4$ on Moyal space

- Infinite volume limit (i.e. $\theta \to \infty$) turns discrete matrix indices into continuous variables $a, b, \dots \in \mathbb{R}_+$ and sums into integrals
- Need energy cutoff $a, b, \dots \in [0, \Lambda^2]$ and normalisation of lowest Taylor terms of two-point function $G_{|nm|} \mapsto G_{ab}$
- Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$, using $G_{b0} = G_{0b}$)

Let $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^{\Lambda^2} \frac{f(p)\,dp}{p-a}$ be the *finite Hilbert transform*.

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$ and G_{a0} solution of

$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Discussion

Together with explicit (but complicated for $G_{ab|cd}$, $G_{ab|cd|ef}$, ...) formulae for higher correlation functions, we have exact solution of $\lambda\phi_4^4$ on extreme Moyal space in terms of

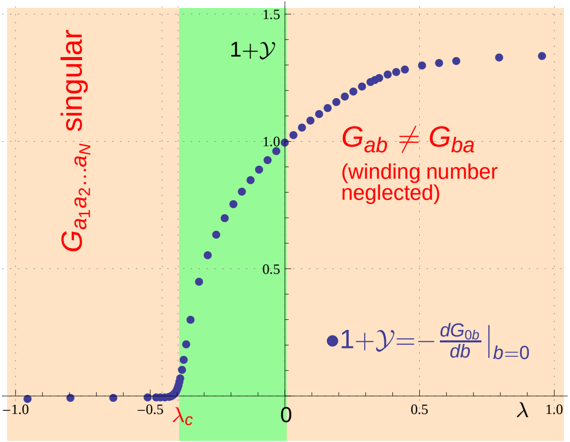
$$G_{b0} = G_{0b} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^{\Lambda^2}\frac{dp}{(\lambda\pi p)^2+\left(t+\frac{1+\lambda\pi p\mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Possible treatments

1. perturbative solution: reproduces all Feynman graphs, generates polylogarithms and ζ-functions
2. iterative solution on computer: nicely convergent
   found interesting phase structure
3. rigorous existence proof of a solution
4. work in progress: (guess); should give uniqueness

Computer simulation: evidence for phase transitions

piecewise linear approximation of G_{0b}, G_{ab} for $\Lambda^2 = 10^7$ and 2000 sample points. Consider $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\big|_{b=0}$



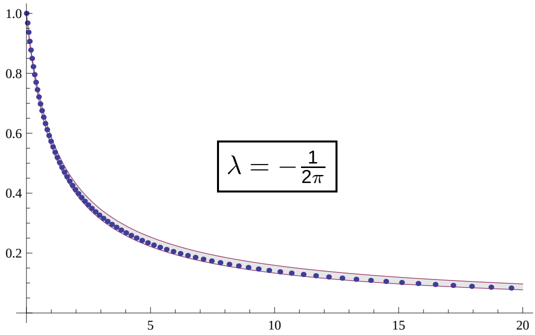
- $(1+\mathcal{Y})'(\lambda)$ discontinuous at $\lambda_c = -0.32$
- A key property for Schwinger functions is realised in $]\lambda_c, 0]$, outside?

Fixed point theorem

Theorem (2015)

Let $-\frac{1}{6} \leq \lambda \leq 0$. Then the equation has a C_0^1-solution

$$\frac{1}{(1+b)^{1-|\lambda|}} \leq G_{0b} \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$$



Proof via Schauder fixed point theorem.

This involves continuity and compactness of a certain operator (in norm topology)

Osterwalder-Schrader reflection positivity

Proposition (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}, \qquad \rho - \text{positive measure}.$$

Excluded for any $\lambda > 0$ (unless rescued by winding number)

- naïve anomalous dimension η positive for $\lambda > 0$,
- renormalisation oversubtracts: η_{ren}, λ of opposite sign
- p-space 2-point function $\frac{1}{(p^2+m^2)^{1-\eta/2}}$
- Need (analytical?) continuation

Reflection positivity simplifies the problem

If G_{x0} is Stieltjes, then Hilbert transform can be avoided:

$$\frac{G_{xy}}{G_{x0}} = \exp\left(-\frac{1}{\pi}\int_1^\infty \frac{dt}{t+x} \operatorname*{arctan}_{[0,\pi]}\left(\frac{y\ \mathrm{Im}(G_{-(t+\mathrm{i}\epsilon),0})}{1-\lambda t\int_0^\infty ds\ \frac{G_{s0}}{t+s} + y\ \mathrm{Re}(G_{-(t+\mathrm{i}\epsilon),0})}\right)\right)$$

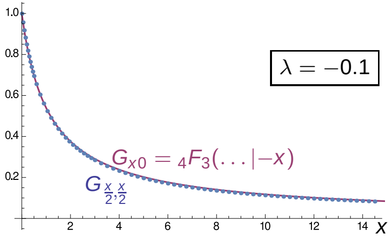
Which class of functions has desired analyticity+holomorphicity and manageable integral transforms?

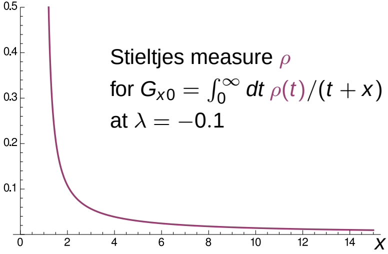
hypergeometric functions $G_{x0} = {}_nF_{n-1}\left(\begin{smallmatrix} a, b_1, \dots b_{n-1} \\ c_1, \dots, c_{n-1}\end{smallmatrix}\Big| -x\right)$ if $a \in [0,1]$ and $c_i > b_i > a$

- holomorphicity at $y > 0$: determine a, b_i, c_i by $G^{(k)}_{0y} = G^{(k)}_{y0}$
- find: $a = 1 + \frac{1}{\pi}\arcsin(\lambda\pi)$, $\prod_{i=1}^n \frac{c_i - 1}{b_i - 1} = \frac{\arcsin(\lambda\pi)}{\lambda\pi}$
- critical coupling constant is $\lambda_c = -\frac{1}{\pi} = -0.3183\dots$

Källén-Lehmann spectrum

- Numerics makes it completely clear (but doesn't prove) that G_{x0} is Stieltjes
- reflection positivity equivalent to G_{xx} a Stieltjes function
- the shape makes this plausible:





- measure for G_{x0} has mass gap $[0,1[$, but no further gap (remnant of UV/IR-mixing)
- absence of the second gap (usually $]1,4[$) circumvents triviality theorems

An analogy

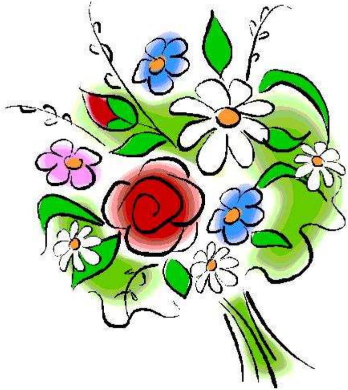
2D Ising model	4D nc ϕ^4-theory
temperature T, $K = \frac{J}{k_B T}$	frequency Ω
Kramers-Wannier duality $\sinh(2K)\sinh(2K^*) = 1$	Langmann-Szabo duality $\Omega\Omega^* = 1$
solvable at $K = K^*$ scale-invariant	solvable at $\Omega = \Omega^*$ almost scale-invariant
CFT minimal model ($m = 3$)	matrix model
operator product expansion Virasoro constraints	Schwinger-Dyson equation Ward identities
critical exponents $G^{\sigma\sigma}_{n0} \propto \frac{1}{n^{d-2+\eta}}$, $\eta = \frac{1}{4}$	critical exponents $G^{\phi\phi}_{n0} \propto \frac{1}{n^{2+\eta}}$, $\lambda \in]\lambda_c, 0]$
Virasoro algebra, CFT, subfactors, ...	**???**

Summary

1. $\lambda\phi^4_4$ on nc Moyal space is, at infinite noncommutativity, exactly solvable in terms of a fixed point problem
   - theory defined by quantum equations of motion (= Schwinger-Dyson equations)
   - existence proved for $-\frac{1}{6}<\lambda\leq 0$
   - phase transitions and critical phenomena
2. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$ = hidden noncommutativity asymptotic safe
   - full Euclidean symmetry (completely unexpected)
   - no momentum exchange
     A consequence of integrability?
   - numerical approach with tiny error: leaves no doubt that Schwinger 2-point function is reflection positive for $-\frac{1}{\pi}<\lambda\leq 0$
3. ready to embark on higher Schwinger functions

Congratulation to your 60 th birthday!

to **VINCENT**



to Marie-France

Hope to keep contacts...