The constructive years of Vincent

1983 – 2005

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IHP. Nov. 25, 2015

Outline

Preliminary

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- **74-77 Ecole Normale Supérieure**: Vincent meets F. David, B. Duplantier, J.-F. Joanny, J.-M. Raymond. He chooses to continue his studies in physics.
- **77-78 DEA of atomic physics,** member of the Centre de Physique Théorique (Ecole polytechnique) where he will remain more than twenty years.
  A. S. Wightman accepts to look after the beginnings of Vincent.
- **78-79 Princeton**: visiting graduate student. He works on Fermionic field theory and with E. Speer on field theory in non-integer dimensions.
- **Euclidean field theory**: all the works of Vincent are in the Euclidean space.

Ph D and post-doc.

- **79-81**: **C. de Calan** accepts to be his Ph D supervisor. He learns perturbation field theory as explained by M. Bergère who was at Saclay (and was F. David supervisor) . He meets there D. Iagolnitzer.
  De Calan and Vincent decide to try and then succeed to prove the local existence of the Borel transform of the **renormalized** perturbation series of φ^4 in four dimensions.
  This amounts to prove "uniform" bounds on the sum of the renormalized graphs of a given order : the bounds of de Calan-Rivasseau.
- **81-83 Princeton**: He writes his thesis and is a docteur d'état in june 82. He meets in Princeton other young people : C. Bachas, E. d'Hoker, A. Sokal, and older ones : F. Dyson and with G. Gallavotti he has numerous discussions on "constructing" φ^4_4.

The de Calan-Rivasseau bounds for φ^4_4

formally: $G(\lambda)(x,y) \;=\; \dfrac{\int d\mu_C \;\; \varphi(x)\varphi(y) \;\; e^{-\lambda\int \varphi^4 + \delta m^2 \int \varphi^2}}{\int d\mu_C \;\; e^{-\lambda\int \varphi^4+\delta m^2 \int \varphi^2}}$

A graph is made of point-like vertices linked by propagators and each propagator between two vertices has a momentum k and an α **parameter** :

$\tilde{C}_k = \frac{1}{k^2+m^2} \;\; = \;\; \int_0^\infty d\alpha \;\, e^{-\alpha(k^2+m^2)}$

formally $\sum_n \sum_{G \;\; order \;\; n} G(\lambda)$; it doesn't make sens because the number of graphs of order n is in $n!$ and because each $G(\lambda)$ is infinite.

formally $\sum_G \;\; G(\lambda) = \;\sum_n \sum_{G \;\; order \;\; n} G_{ren}(\lambda_0)$

where $G_{ren}(\lambda_0)$is **finite** and is the renormalized graph of G,
λ_0 is the renormalized coupling

$|\sum_{G \;\; of \;\; order \;\; n} \;\; G_{ren}(\lambda_0)| \;\; \leq n! \; (cst)^n \lambda_0^n$ is the **de Calan-Rivasseau bound**

Borel sum: $\;\sum_n \sum_{G \;\; of \;\; order \;\; n} \; \frac{1}{n!} \; G_{ren}(\lambda_0)$ finite radius of convergence in λ_0 .

A renormalized graph is given by the complicated Zimmermann's formula.
The proof of de Calan and Rivasseau was a "tour de force" in Zimmermann's like things.

The constructive years

1. Preliminary

2. The constructive years
   - Pointlike singularities: perturbation, φ^4_4, Gross-Neveu
   - Intermediate years
   - Many Fermions systems

3. Students

4. Beyond local QFT

Renormalizable models

Construction means to prove the existence of the limit of the correlation functions of a just renormalizable field theory model as the cutoff is removed.

- **Autumn 83**: Vincent is back in Paris and intends to construct a just renormalizable model
  J. Feldman is there on a sabbatical, local team (**R. Sénéor**, J. M.)
- **85** **a multiscale proof** of the de Calan Rivasseau bounds
- **86** Construction of the massive **Gross-Neveu model**
- **87** Construction of φ_4^4 of **mass zero** with an UV cutoff

Those works belong to the Glimmm-Jaffe-Spencer school and they were done in parallel to the works of the roman team (G. Gallavotti) and of K. Gawedzki and A. Kupiainen which were very close to the Block spin approach. All these things can be called rigourous renormalization group.

- **85-87** bounds on **large orders**: **F. David, F. Nicolo**
- **93** Construction of $\mathbf{YM}_4$ in a finite volume : difficult, to be done better.

regularized diagrams of φ^4_4 and effective couplings

$$\textit{Scales}: \quad C_{x,y} = \int_0^\infty d\alpha \; C^\alpha_{x,y}, \quad C^\alpha_{x,y} = \frac{e^{-\alpha m^2}}{(4\pi)^2\alpha} \; \frac{e^{-(x-y)^2/\alpha}}{\alpha} \tag{1}$$

$\frac{1}{|x_\ell - y_\ell|} \sim \alpha_\ell^{-1/2}$ is the scale of the propagator ℓ

G_ℓ connected subgraph containing ℓ :
internal links : connected to ℓ and scales $\geq \alpha_\ell^{-1/2}$
external links : hooked to G_ℓ and scales $< \alpha_\ell^{-1/2}$, number $\mathbf{e}_\ell$

extract a tree with the **largest scales** for the links between vertices
(i.e. $\tilde{C}(k)$'s as small as possible)
If the labelling is s.t. :$0 < \alpha_1 < \alpha_2 < \ldots$

$$\int \{\tfrac{d\alpha}{\alpha}\} \; |G(\{\alpha\},\lambda)| \leq \lambda^{|G|} \sum_{labelling} \int_{0 \leq \alpha_1 \leq \alpha_2 \leq \ldots} \prod_\ell \tfrac{d\alpha_\ell}{\alpha_\ell} \; e^{-\alpha_\ell m^2} \big(\tfrac{\alpha_\ell}{\alpha_{\ell+1}}\big)^{(e_\ell - 4)/2}.$$

$(e_\ell - 4)/2$ doesn't depend on the number of vertices : **renormalisability**.
If some $e_\ell \leq 4$ the integral is divergent in the region $\alpha_\ell^{-1/2} \to \infty$, of the large scales : it is the UV divergence.

all $e_\ell - 4 > 0$ completely convergent graphs.

This validates the choice of the tree of highest scales.

If $e_\ell = 4$: the subgraph is a contribution to the **effective coupling**.

$$(4 \text{ point subgraph}) = \text{ local part } \quad + \quad \textit{remainder}$$

the local part is added to the coupling constant $\Rightarrow$ each vertex v has a running or effective coupling constant λ_{scale_v} a function of the biggest scale hooked to v.

$|$remainder$|$ bound with $\left(\frac{\alpha_\ell}{\alpha_{\ell+1}}\right)^{(e_\ell+1-4)/2}$ "has" 5 external legs: it is **regularized**.

A completely regularized diagram G_{reg} is then convergent.

At order n the number of diagrams is $\sim n!$ (**instantons**)

$$\sum_G G \leq \int \sum_G \left(\prod_\ell \frac{d\alpha_\ell}{\alpha_\ell}\right) G_{reg}(\{\alpha\},\ \{\lambda_{scale_v}\})$$

$$\sum_{G\ni x,\ order\ \ n} \int \left(\prod_\ell \frac{d\alpha_\ell}{\alpha_\ell}\right) |G_{reg}(\{\alpha\},\ 1)| \leq (n!)(cst)^n$$

The **effective parameters** λ_{scale} (and others) give the **leading** behaviors. .

Note : expressing λ_{scale_v} as a power series in $\lambda_{(scale\ 0)}$ gives the **renormalized** perturbation expansion thus $\lambda_{ren} = \lambda_0$.

Renormalization

Let $\int dx_1 ... dx_4 \ G^{(scale)}_{x_1,...,x_4} \ C^{(ext.scale)}_{x_1,z_1} ... C^{(ext.scale)}_{x_4,z_4}$ be a 4-point subgraph $(scale) > (ext.scale)$

The local part

$\int dx_2 ... dx_4 \ G^{(scale)}_{x_1,...,x_4} \qquad \int dx_1 \Big(\prod C^{(ext.scale)} \text{ hooked to } x_1 \Big)$

$\int dx_2 ... dx_4 \ G^{(scale)}_{x_1,...,x_4} \ = \ \tilde{G}^{(scale)}|_{\text{zero momenta}} \ = \ \frac{1}{(scale)}(\beta \ function)_{(scale)}$

this gives $\frac{d\lambda_{(scale)}}{d(scale)} = \frac{1}{(scale)}(\beta \ function)_{(scale)} \Rightarrow$ the flow of the effective coupling

The regularized part has one external propagator like:

$$C^{(ext.scale)}_{x_2,z_2} - C^{(ext.scale)}_{x_1,z_2} \simeq (x_2 - x_1) \nabla C^{(ext.scale)}_{x_2,z_2} \sim \frac{(ext.scale)}{(scale)} \ \ C^{(ext.scale)}$$

using the behaviors of $G^{(scale)}, C^{(ext.scale)}$. Thus the regularized term has a power counting as if it had one more external propagator.

Constructive field theory

Using the material above to obtain a non perturbative construction of a renormalizable model:

- the model must be such that, based on first orders perturbations, all the effective couplings λ_{scale} are small enough.

-extracting a tree of connections of the highest scales : multiscale expansion

- regularizing all 2, 4 point subgraphs

- for each tree obtaining a bound which is proportionnal to the bound of **one** regularized perturbative graph (having the same tree) ; this is the core of the constructive part.

Then the expansion converges all coupling being small.

Constructive for Fermions

Gaussian Fermionic integral with propagator S, $S_{x,y} = \int dz\ A_{x,z} B_{z,y}$

$\int d\mu(\psi,\bar\psi)\ \psi_{x_1}...\psi_{x_n}\bar\psi_{y_1}...\bar\psi_{y_n} = det\Big(S_{x_i,y_j}\Big)$, the Gram's bound gives :

$|det\Big(S_{x_i,y_j}\Big)| \le \prod_j ||A_{x_j,.}||_2\ ||B_{y_j,.}||_2$: no factorials.

massive Gross-Neveu : a quartic interaction in two dimensions, just renormalizable and asymptotically free.
$(\beta\ function) < 0$ so that

$0 = \lambda_\infty < \lambda_{(scale)} < \lambda_0 << 1$ (asymptotic freedom)

The sum of graphs having the same tree : the propagators not on the tree form a determinant which gives no factorial and the Gram's bound reproduces the perturbative power counting (non perturbative bound).

The number of trees connecting n vertices yields $\sim n!$ it is compensated by the $1/n!$ from the exponential.

Thus the expansion is convergent for λ_0 small.

Note: the Taylor expansion in λ_0 (the renormalized series) is not analytic because the expansion of the $\lambda_{(scale)}$ in λ_0 is logarithmically divergent.

Constructive for Bosons: $\lambda\varphi_4^4$

Bosons don't give determinants, the zero mass model is asymptotically free as $(scale) \to 0$. Thus we take $0 \leq (scale) \leq 1$ and λ small. The idea is to extract the multiscale tree between the degree of freedom (and not between the vertices) and then to analyse each degree of freedom non perturbatively.

With $M > 1$ we define the scale k to correspond to $M^{-k} < \alpha^{-1/2} \leq M^{-(k-1)}$

$$C^k_{x,y} \sim M^{-2k+2}\ e^{-[M^{-k}|x-y|]^2}$$

scale decomposition: $\varphi = \sum_k \varphi^k$ where φ^k of scale k

the degrees of freedom $(\Delta x) \times \Delta(scale) \sim 1$:

For each scale k a lattice of cubes Δ_k of side M^k.

$$\varphi^k(x) = \sum_{\Delta_k} \varphi^k(x)|_{x\in\Delta_k}$$

each Δ_k symbolizes one degree of freedom. All the fields $\varphi^k(x)$ with $x \in \Delta_k$ are "localized in Δ_k.

two degree of freedom Δ_k and $\Delta'_{k'}$ can be linked by:

if $k = k'$ **horizontal** link: propagator C^k between two vertices in the two degrees of freedom

if $k > k'$ and if $\Delta_k \subset \Delta'_{k'}$ **vertical** link: vertex $\varphi^4(x),\ x \in \Delta_k$ having fields in the two degrees of freedom. For the subgraph G^k containing Δ_k the fields of $\Delta_{k'}$ connected to it are external field. Then one is led to expand vertically further in order to factorize the 2 or 4 point function or generate at least 5 external legs per subgraph.

A two point function is formally: $\dfrac{\int d\mu_C(\varphi)\ \varphi(x)\varphi(y)\quad e^{-\int \lambda\varphi^4 + \int \delta m^2 \varphi^2}}{\int d\mu_C(\varphi)\ e^{-\int \lambda\varphi^4 + \int \delta m^2 \varphi^2}}$

The two integrals are expanded independently in product of **non overlapping** perturbative multi scale "trees" ; there is, one propagator or at most 5 vertices by linked localization.
Computing the ratio in terms of trees can be done if the sum of connected trees containing one given localization is $\leq (small\ \ factor)^{\#(degree\ \ of\ \ freedom\ \ in\ \ the\ \ tree)}$

This expansion, at small λ converges because there is (morally!) a bounded number of fields per localization produced by the perturbative expansion.

If the link is a propagator C^k between Δ_k and Δ'_k it generates fields (roughly) of scales lower than k i.e. $\varphi \simeq \frac{1}{|\Delta_k|}\int_{\Delta_k}\varphi$
, we "dominate" them with the exponential of the interaction, for example :

$$\varphi^p \, e^{-\int_{\Delta_k} dx \;\; \lambda\varphi^4} \leq (p/4)! \, \left(\lambda^{-1/4}(\frac{1}{|\Delta_k|})^{1/4}\right)^p \;=\; (p/4)! \, \left(\lambda^{-1/4}\, M^{-k}\right)^p$$

There is at most 3 fields to dominate per vertex so it remains a small factor $\lambda^{1/4}$ per vertex. The factor M^{-k} per field corresponds to the power counting of a field of scale k.

These bounds are the non-perturbative part of the argument.

The bound for a contribution of a tree is similar to a perturbative bound on a corresponding graph.

The expansion converges at small λ because then all $\lambda_{scale} \leq \lambda$
in fact $\lambda_{scale} \to 0 \; as \;\; scale \to 0$.

Intermediate years

- **89, 92 J. de Coninck, F. Dunlop, P. Roche** JM: interface in statistical mechanics
- **86, 87, 91 D. Arnaudon, C. Bachas, P. Végreville** and polytechnique students: strings
- **91** matrix models at given genus: interaction with local mathematicians
- **95 C. Kopper** JM: mass generation at large N for the Gross-Neveu model

Many Fermions systems

J. Feldman and E. Trubowitz, JM

Joel and Eugene began first to study perturbatively simplified many Fermions system where the infrared singularity is on a circle or a sphere and the interaction is local. Then we discussed together how to developp a multiscale approach.

- **92** multiscale expansion for a model with singularity on a **Fermi surface**
- **93** Ward identities for a U(1) Goldstone Boson
- **93** 2d models as **vector like models**. A dynamical $1/N$ argument :
- **95** a **one scale** expansion for the **3 d** case

Many Fermions systems in $2+1$ dimensions

The simplest model: the **Jellium** : local quartic interaction and propagator with singularity at : $p_0 = 0$ and $|\mathbf{p}| = \sqrt{2m\mu}$ (Fermi surface).

It is a renormalizable model.

We fix an UV cutoff and look at the infrared behavior.

the *kth* momentum scale : $M^{-2k} \leq p_0^2 + (\frac{\mathbf{p}^2}{2m} - \mu)^2 < M^{-2(k-1)}$
it is a shell of thickness of order $M^{-(k-1)}$ around the circular singularity.

the model in the *kth* scale
at each vertex the sum over the momenta of the four propagators are four vectors (almost) on the singular circle and their sum is zero : the momenta are on an approximate rhombus : each vertex is characterized by the angle of the rhombus.

we cut the shell in sectors of width M^{-k} (more complicated)

momentum = (sector-vector "on" the Fermi circle) + (a local vector $\sim M^{-k}$)

At each vertex sector-vectors are on a rhombus with a discrete angle (M^k values).

Extracting a tree, fixing the sectors not on the tree fixes all the sectors.
All the propagators not on the tree and in the same sector form a determinant.
There is a sum on sectors for each vertex. One recovers for a graph the perturbative power counting.
$\Rightarrow$ the perturbation series, in the *kth* scale, converges at small coupling.

To have a multiscale expansion it remains to regularise the 2, 4 points subgraphs.

renormalization:

in φ^4 subtracting at zero momentum   e.g. subdiagram's external momenta on the singularity.
Here we subtract the value for external propagators hooked at the same point and **with momenta on the Fermi surface**.

The only vertices which flow correspond to approximate Cooper pairs : collapsed rhombus of sectors.

taking λ_1 small, the effective coupling of the approximate Cooper pairs are increasing logarithmically ; the running coupling λ_k will remain small until the scales : $M^{-k} \sim e^{-1/\lambda}$.

Students

Students years in Palaiseau

1995 – 2001

- **97, 98 A. Abdesselam** : multiscale expansion, cluster expansion à la Brydges-Kennedy (BKAR), Tree expansion for Fermionic models
- **98 G. Poirot**,JM : 2d Anderson model at weak disorder ; circular singularity, not renormalizable ; study above the critical point uses some kind of limited $1/N$ expansion and Ward identities.
  2003, J. Bellissard, JM: 2d Anderson supersymmetric as a matrix model.
- **2000 M. Disertori** : Fermionic expansion for the Jellium model with continuous scale parameters, Fermi liquid behavior for the Jellium model in 2d and in 2011,2013 for 3d (JM)

Students years in Orsay

Since 2002

- **2002:** The Hubbard model at half filling: the Fermi surface is a square ; number of sectors logarithmic in the scale and the sectors mainly in "corners" of the momentum shells. Convergent contributions
- **2005 S. Afchain**, JM: renormalization ; lower bound on the self energy : the model is not a Fermi liquid

Beyond local QFT

Beyond local quantum field theory

2004 – present

Research interests:

- **non-commutative field theories**
- **topological graph polynomials**
- **group field theory, tensor field theories**
- **loop vertex expansion**

with many students and some collaborators: **R.C. Avohou, P. Beliavski, J. Ben Geloun, V. Bonzom, S. Carrozza, S. Dartois, T. Delepouve, M. Disertori, R. Gurau, T. Krajewski, V. Lahoche, L. Lionni, A.P.C. Malbouisson, K. Nouy, D. Oriti, A. Riello, M. Smerlak, A. Tanasa, F. Vignes-Tourneret, P. Vitale, R. Wulkenhaar, Zhituo Wang** , JM

Loop Vertex Expansion

2007 – present

first for matrices and developped later with R. Gurau, Zhituo Wang , JM

intermediate field σ : $e^{-\lambda\varphi^4/2} = \frac{1}{\sqrt{2\pi}} \int d\sigma e^{i\sqrt{\lambda}\varphi^2\sigma \ -\sigma^2/2}$ $\Rightarrow$ **quadratic in** φ

$Z = \int d\mu_\delta(\sigma)\ e^{(-1/2)Tr\ ln(1+2iC^{1/2}\sqrt{\lambda}\sigma C^{1/2})} = \ e^{\sum \text{connected graph}}$

the vertices are (non-perturbative) loops, the propagators are δ functions. Each graph is expanded as a sum over trees of loops.

$$\text{loop}(x) = -2iC^{1/2}(x,.)\frac{1}{1+2iC^{1/2}\sqrt{\lambda}\sigma C^{1/2}}C^{1/2}(.,x)$$

$||\frac{1}{1+2iC^{1/2}\sqrt{\lambda}\sigma C^{1/2}}|| \leq 1 \Rightarrow |\text{loop}(x)| \leq 2||C^{1/2}(x,.)||_2^2$

Each new loop is like the insertion of the operator $C^{1/2}[loop]C^{1/2}$ which norm is $\leq (cst)\lambda$ for φ_4^4 in a single scale.

The loop expansion is convergent in **one scale** for φ_4^4 at small coupling. It opens the way to simplifications of proofs or further new non perturbative results.