Some wonderful conjectures (but very few theorems) at the boundary between analysis, combinatorics and probability

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References:

1. Roots of a formal power series $f(x,y) = \sum_{n=0}^{\infty} a_n(y)\, x^n$, with applications to graph enumeration and q-series,
   Series of 4 lectures at Queen Mary (London), March–April 2011,
   `http://www.maths.qmw.ac.uk/~pjc/csgnotes/sokal/`

2. The leading root of the partial theta function, arXiv:1106.1003 [math.CO], Adv. Math. **229**, 2603–2621 (2012).

Vincent has always been a master of combinatorics ...

The multiple forest formula states:

Theorem IV.1 (The Algebraic Brydges-Kennedy Jungle Formula)

$$\exp\Big(\sum_{\substack{l\in\mathcal{P}_n\\1\le k\le m}} u_l^k\Big) = \sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\dots,\mathfrak{F}_m)\\ m-\text{jungle}}} \Big(\prod_{l\in\mathfrak{F}_m}\int_0^1 dh_l\Big)\Big(\prod_{k=1}^m\Big(\prod_{l\in\mathfrak{F}_k\setminus\mathfrak{F}_{k-1}} u_l^k\Big)\Big)$$

$$.\ \exp\Big(\sum_{k=1}^m\sum_{l\in\mathcal{P}_n} h_l^{\mathcal{F},k}(\mathbf{h}).u_l^k\Big) \qquad \text{(IV.1)}$$

where $\mathfrak{F}_0=0$ by convention, $\mathbf{h}$ is the vector $(h_l)_{l\in\mathfrak{F}_m}$ and the functions $h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h})$ are defined in the following manner:
- If i and j are not connected by $\mathfrak{F}_k$ let $h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h})=0$.
- If i and j are connected by $\mathfrak{F}_k$ but not by $\mathfrak{F}_{k-1}$ let

$$h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h}) = \inf\Big\{h_l, l\in L_{\mathfrak{F}_k}\{ij\}\cap(\mathfrak{F}_k\setminus\mathfrak{F}_{k-1})\Big\}$$

(recall that $L_{\mathfrak{F}}\{ij\}$ is the unique path in the forest $\mathfrak{F}$ connecting i to j).
- If i and j are connected by $\mathfrak{F}_{k-1}$ let $h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h})=1$.

Proof: By induction. The case $m=1$ was treated in Section II. For the induction step from m to $m+1$, we sum over the last forest $\mathfrak{F}_{m+1}$:

$$\sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\dots,\mathfrak{F}_{m+1})\\(m+1)-\text{jungle}}}\Big(\prod_{l\in\mathfrak{F}_{m+1}}\int_0^1 dh_l\Big)\Big(\prod_{k=1}^{m+1}\Big(\prod_{l\in\mathfrak{F}_k\setminus\mathfrak{F}_{k-1}} u_l^k\Big)\Big)\exp\Big(\sum_{k=1}^{m+1}\sum_{l\in\mathcal{P}_n} h_l^{\mathcal{F},k}(\mathbf{h}).u_l^k\Big)$$

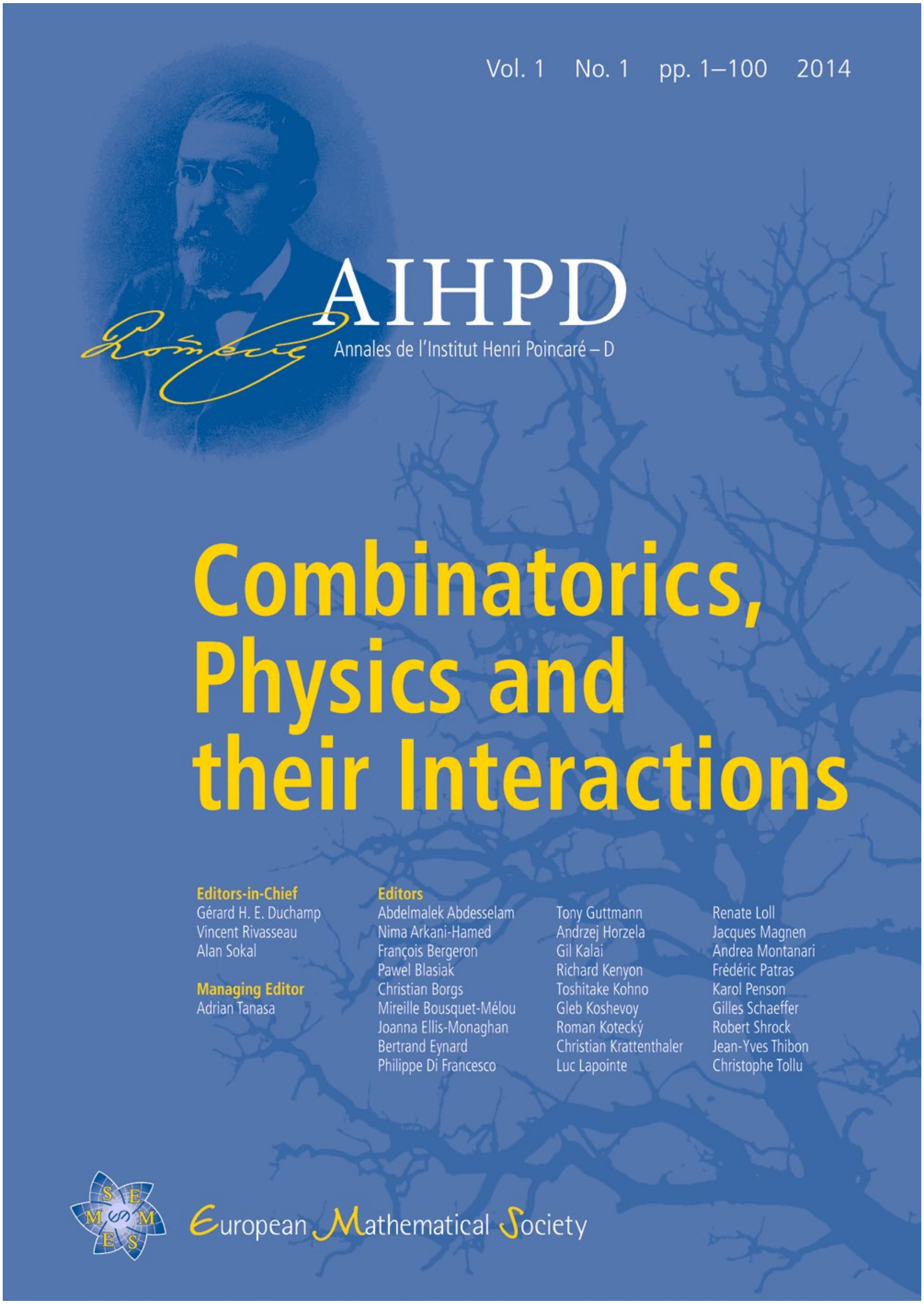
$$= \sum_{\substack{\mathcal{F}'=(\mathfrak{F}_1,\dots,\mathfrak{F}_m)\\m-\text{jungle}}}\Big(\prod_{l\in\mathfrak{F}_m}\int_0^1 dh_l\Big)\Big(\prod_{k=1}^{m}\Big(\prod_{l\in\mathfrak{F}_k\setminus\mathfrak{F}_{k-1}} u_l^k\Big)\Big)\exp\Big(\sum_{k=1}^{m}\sum_{l\in\mathcal{P}_n} h_l^{\mathcal{F}',k}(\mathbf{h}').u_l^k\Big)$$

$$.\ \sum_{\substack{\mathfrak{F}_{m+1}\\\mathfrak{F}_m\subset\mathfrak{F}_{m+1}}}\Big(\prod_{l\in\mathfrak{F}_{m+1}\setminus\mathfrak{F}_m}\int_0^1 dh_l\Big)\Big(\prod_{l\in\mathfrak{F}_{m+1}\setminus\mathfrak{F}_m} u_l^{m+1}\Big)\exp\Big(\sum_{l\in\mathcal{P}_n} h_l^{\mathcal{F},m+1}(\mathbf{h}).u_l^{m+1}\Big)$$

(IV.2)

where $\mathbf{h}=(h_l)_{l\in\mathfrak{F}_{m+1}}$, $\mathbf{h}'=(h_l)_{l\in\mathfrak{F}_m}$ and we have noted that if $1\le k\le m$ then $h_l^{\mathcal{F},k}(\mathbf{h})=h_l^{\mathcal{F}',k}(\mathbf{h}')$. To perform the summation over $\mathfrak{F}_{m+1}$, we will use the forest formula of section II and our favorite argument of forgetting the details of the tree structure up to $\mathfrak{F}_m$, to concentrate on what $\mathfrak{F}_{m+1}$ brings as new connections between the existing clusters. We introduce the partition $\mathcal{D}$ of I_n created by $\mathfrak{F}_m=\{l_1,\dots,l_\nu\}$ and the u-forest on $\mathcal{D}$, $\overline{\mathfrak{F}}_{m+1}=\{\bar{l}_1,\dots,\bar{l}_{\tau-\nu}\}$ induced by $\mathfrak{F}_{m+1}\setminus\mathfrak{F}_m=\{l_{\nu+1},\dots,l_\tau\}$ with $\nu\le\tau$. The definitions are the same as in the proof of Lemma II.2 except that we have u-forests instead of o-forests.

For a link $\{ab\}$ between two elements a and b of $\mathcal{D}$, let $\overline{u}_{\{ab\}}=\sum_{i\in a,j\in b}u^{m+1}_{\{ij\}}$. Summing over $\mathfrak{F}_{m+1}$, u-forest on I_n containing $\mathfrak{F}_m$, with the "propagators" $u^{m+1}_{\{ij\}}$



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The deformed exponential function $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$

- Defined for complex x and y satisfying $|y| \le 1$
- Analytic in $\mathbb{C} \times \mathbb{D}$, continuous in $\mathbb{C} \times \overline{\mathbb{D}}$
- $F(\,\cdot\,, y)$ is entire for each $y \in \overline{\mathbb{D}}$
- Valiron (1938): "from a certain viewpoint the simplest entire function after the exponential function"

Applications:

- Statistical mechanics: Partition function of one-site lattice gas
- Combinatorics: Enumeration of connected graphs,
  generating function for Tutte polynomials on K_n
  (also acyclic digraphs, inversions of trees, ...)
- Functional-differential equation: $F'(x) = F(yx)$ where $' = \partial/\partial x$
- Complex analysis: Whittaker and Goncharov constants

Application to enumeration of connected graphs

- Let $a_{n,m}$ = # graphs with n labelled vertices and m edges
- Generating polynomial $A_n(v) = \sum\limits_m a_{n,m}\, v^m$
- Exponential generating function $A(x,v) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} A_n(v)$
- Of course $a_{n,m} = \binom{n(n-1)/2}{m} \quad \Longrightarrow \quad A_n(v) = (1+v)^{n(n-1)/2} \quad \Longrightarrow$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} A_n(v) \;=\; F(x, 1+v)$$

- Now let $c_{n,m}$ = # *connected* graphs with n labelled vertices and m edges
- Generating polynomial $C_n(v) = \sum\limits_m c_{n,m}\, v^m$
- Exponential generating function $C(x,v) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n!} C_n(v)$
- No simple explicit formula for $C_n(v)$ is known, but ...
- The *exponential formula* tells us that $C(x,v) = \log A(x,v)$, i.e.

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} C_n(v) \;=\; \log F(x, 1+v)$$

  [see Tutte (1967) and Scott–A.D.S., arXiv:0803.1477 for generalizations to the Tutte polynomials of the complete graphs K_n]

- Usually considered as formal power series
- But series are *convergent* if $|1+v| \le 1$
  [see also Flajolet–Salvy–Schaeffer (2004)]

Elementary analytic properties of $F(x,y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$

- $\boldsymbol{y=0}$: $F(x,0) = 1 + x$

- $\boldsymbol{0 < |y| < 1}$: $F(\,\cdot\,,y)$ is a nonpolynomial entire function of order 0:

$$F(x,y) \;=\; \prod_{k=0}^{\infty} \left(1 \,-\, \frac{x}{x_k(y)}\right)$$

  where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 0$

- $\boldsymbol{y=1}$: $F(x,1) = e^x$

- $\boldsymbol{|y| = 1}$ **with** $\boldsymbol{y \neq 1}$: $F(\,\cdot\,,y)$ is an entire function of order 1 and type 1:

$$F(x,y) \;=\; e^x \prod_{k=0}^{\infty} \left(1 \,-\, \frac{x}{x_k(y)}\right) e^{x/x_k(y)} \,.$$

  where $\sum |x_k(y)|^{-\alpha} < \infty$ for every $\alpha > 1$

  [see also Ålander (1914) for y a root of unity; Valiron (1938) and Eremenko–Ostrovskii (2007) for y not a root of unity]

- $\boldsymbol{|y| > 1}$: The series $F(\,\cdot\,,y)$ has radius of convergence 0

Consequences for $C_n(v)$

- Make change of variables $y = 1 + v$:

$$\overline{C}_n(y) \;=\; C_n(y-1)$$

- Then for $|y| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \overline{C}_n(y) \;=\; \log F(x,y) \;=\; \sum_k \log\Bigl(1 - \frac{x}{x_k(y)}\Bigr)$$

  and hence

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n} \qquad \text{for all } n \ge 1$$

  (also holds for $n \ge 2$ when $|y| = 1$)

- This is a *convergent* expansion for $\overline{C}_n(y)$
- In particular, gives large-n asymptotic behavior

$$\overline{C}_n(y) \;=\; -(n-1)!\, x_0(y)^{-n} \left[1 + O(e^{-\epsilon n})\right]$$

  whenever $F(\,\cdot\,, y)$ has a unique root $x_0(y)$ of minimum modulus

Question: What can we say about the roots $x_k(y)$?

Small-y expansion of roots $x_k(y)$

- For small $|y|$, we have $F(x,y) = 1 + x + O(y)$, so we expect a convergent expansion

$$x_0(y) \;=\; -1 \;-\; \sum_{n=1}^{\infty} a_n y^n$$

(easy proof using Rouché: valid for $|y| \lesssim 0.441755$)

- More generally, for each integer $k \ge 0$, write $x = \xi y^{-k}$ and study

$$F_k(\xi,y) \;=\; y^{k(k+1)/2} F(\xi y^{-k}, y) \;=\; \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\, y^{(n-k)(n-k-1)/2}$$

Sum is dominated by terms $n = k$ and $n = k+1$; gives root

$$x_k(y) \;=\; -(k+1)y^{-k} \left[1 + \sum_{n=1}^{\infty} a_n^{(k)} y^n \right]$$

Rouché argument valid for $|y| \lesssim 0.207875$ uniformly in k:
all roots are simple and given by convergent expansion $x_k(y)$

- Can also use theta function in Rouché (Eremenko)

Might these series converge for all $|y| < 1$?

Two ways that $x_k(y)$ could fail to be analytic for $|y| < 1$:

1. Collision of roots ($\rightarrow$ branch point)
2. Root escaping to infinity

Theorem (Eremenko): No root can escape to infinity for y in the open unit disc $\mathbb{D}$ (except of course at $y = 0$).

In fact, for any compact subset $K \subset \mathbb{D}$ and any $\epsilon > 0$, there exists an integer k_0 such that for all $y \in K \setminus \{0\}$ we have:

(a) The function $F(\,\cdot\,, y)$ has exactly k_0 zeros (counting multiplicity) in the disc $|x| < k_0 |y|^{-(k_0 - \frac{1}{2})}$, and

(b) In the region $|x| \geq k_0 |y|^{-(k_0 - \frac{1}{2})}$, the function $F(\,\cdot\,, y)$ has a simple zero within a factor $1 + \epsilon$ of $-(k+1)y^{-k}$ for each $k \geq k_0$, and no other zeros.

- Proof is based on comparison with a theta function (whose roots are known by virtue of Jacobi's product formula)
- *Conjecture* that roots cannot escape to infinity even in the *closed* unit disc except at $y = 1$

Big Conjecture #1. All roots of $F(\,\cdot\,, y)$ are simple for $|y| < 1$. [and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #1. Each root $x_k(y)$ is analytic in $|y| < 1$.

But I conjecture more ...

Big Conjecture #2. The roots of $F(\,\cdot\,, y)$ are non-crossing *in modulus* for $|y| < 1$:

$$|x_0(y)| \;<\; |x_1(y)| \;<\; |x_2(y)| \;<\; \ldots$$

[and also for $|y| = 1$, I suspect]

Consequence of Big Conjecture #2. The roots are actually separated in modulus by a factor at least $|y|$, i.e.

$$|x_k(y)| \;<\; |y|\,|x_{k+1}(y)| \qquad \text{for all } k \geq 0$$

Proof. Apply the Schwarz lemma to $x_k(y)/x_{k+1}(y)$.

Consequence for the zeros of $\overline{C}_n(y)$

Recall

$$\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n}$$

and use a variant of the Beraha–Kahane–Weiss theorem [A.D.S., arXiv:cond-mat/0012369, Theorem 3.2] $\Longrightarrow$ the limit points of zeros of $\overline{C}_n$ are the values y for which the zero of minimum modulus of $F(\,\cdot\,, y)$ is *nonunique*.

So if $F(\,\cdot\,, y)$ has a *unique* zero of minimum modulus for all $y \in \mathbb{D}$ (a weakened form of Big Conjecture #2), then the zeros of $\overline{C}_n$ do not accumulate anywhere in the open unit disc.

I actually conjecture more (based on computations up to $n \approx 80$):

Big Conjecture #3. For each n, $\overline{C}_n(y)$ has no zeros with $|y| < 1$. [and, I suspect, no zeros with $|y| = 1$ except the point $y = 1$]

What is the evidence for these conjectures?

Evidence #1: Behavior at real y.

Theorem (Laguerre): For $0 \le y < 1$, all the roots of $F(\,\cdot\,, y)$ are simple and negative real.

Corollary: Each root $x_k(y)$ is analytic in a complex neighborhood of the interval $[0, 1)$.

[Real-variables methods give further information about the roots $x_k(y)$ for $0 \le y < 1$: see Langley (2000).]

Now combine this with

Evidence #2: From numerical computation of the series $x_k(y)$ ...

Let MATHEMATICA run for a weekend ...

$$
\begin{aligned}
-x_0(y) \;=\; & 1 + \tfrac{1}{2}y + \tfrac{1}{2}y^2 + \tfrac{11}{24}y^3 + \tfrac{11}{24}y^4 + \tfrac{7}{16}y^5 + \tfrac{7}{16}y^6 \\
& + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\
& + \tfrac{170921}{414720}y^{12} + \tfrac{340171}{829440}y^{13} + \tfrac{22565}{55296}y^{14} \\
& + \ldots + \text{terms through order } y^{899}
\end{aligned}
$$

and all the coefficients (so far) are nonnegative!

- Very recently I have computed $x_0(y)$ through order y^{16383}.
- I also have shorter series for $x_k(y)$ for $k \geq 1$.

Big Conjecture #4. For each k, the series $-x_k(y)$ has all nonnegative coefficients.

Combine this with the known analyticity for $0 \leq y < 1$, and Pringsheim gives:

Consequence of Big Conjecture #4. Each root $x_k(y)$ is analytic in the open unit disc.

But more is true . . .

Look at the *reciprocal* of $x_0(y)$:

$$
\begin{aligned}
-\frac{1}{x_0(y)} = 1 &- \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\
&- \tfrac{1}{96}y^7 - \tfrac{7}{768}y^8 - \tfrac{49}{6912}y^9 - \tfrac{113}{23040}y^{10} - \tfrac{17}{4608}y^{11} \\
&- \tfrac{293}{92160}y^{12} - \tfrac{737}{276480}y^{13} - \tfrac{3107}{1658880}y^{14} \\
&- \ldots - \text{terms through order } y^{899}
\end{aligned}
$$

and all the coefficients (so far) beyond the constant term are *nonpositive*!

Big Conjecture #5. For each k, the series $-(k+1)y^{-k}/x_k(y)$ has all *nonpositive* coefficients after the constant term 1.

[This implies the preceding conjecture, but is stronger.]

- Relative simplicity of the coefficients of $-1/x_0(y)$ compared to those of $-x_0(y)$ $\longrightarrow$ simpler combinatorial interpretation?
- Note that $x_k(y) \to -\infty$ as $y \uparrow 1$ (this is fairly easy to prove). So $1/x_k(y) \to 0$. Therefore:

Consequence of Big Conjecture #5. For each k, the coefficients (after the constant term) in the series $-(k+1)y^{-k}/x_k(y)$ are the *probabilities* for a positive-integer-valued random variable.

What might such a random variable be???
Could this approach be used to *prove* Big Conjecture #5?

(see also the next two slides)

But I conjecture that even more is true ...

Define $\overline{C}_n^\star(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$ and recall that $-x_0(y) = \lim_{n\to\infty} \overline{C}_n^\star(y)^{-1/n}$

Big Conjecture #6. For each n,

(a) the series $\overline{C}_n^\star(y)^{-1/n}$ has all nonnegative coefficients,

and even more strongly,

(b) the series $\overline{C}_n^\star(y)^{1/n}$ has all nonpositive coefficients after the constant term 1.

Since $\overline{C}_n^\star(y) > 0$ for $0 \le y < 1$, Pringsheim shows that Big Conjecture #6a implies Big Conjecture #3:

For each n, $\overline{C}_n(y)$ has no zeros with $|y| < 1$.

Moreover, Big Conjecture #6b $\Longrightarrow$ for each n, the coefficients in the series $1 - \overline{C}_n^\star(y)^{1/n}$ are the *probabilities* for a positive-integer-valued random variable.

Such a random variable would generalize the one for $-1/x_0(y)$ in roughly the same way that the binomial generalizes the Poisson.

What might such a random variable be?

- Probability generating function $P_n(y) = 1 - \overline{C}_n^\star(y)^{1/n}$
  where $\overline{C}_n^\star(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$
- Presumably has something to do with random graphs on n vertices
- Maybe some structure built on top of a random graph
  (some kind of tree? Markov chain?)

Try to understand the first two cases:

$$\begin{aligned} P_2(y) &= 1 - (1-y)^{1/2} \\ &= \tfrac{1}{2}y + \tfrac{1}{8}y^2 + \tfrac{1}{16}y^3 + \tfrac{5}{128}y^4 + \tfrac{7}{256}y^5 + \tfrac{21}{1024}y^6 \\ &\qquad + \tfrac{33}{2048}y^7 + \tfrac{429}{32768}y^8 + \tfrac{715}{65536}y^9 + \tfrac{2431}{262144}y^{10} + \ldots \\ &\sim \text{Sibuya}(\tfrac{1}{2}) \text{ random variable} \end{aligned}$$

$$\begin{aligned} P_3(y) &= 1 - (1 - \tfrac{3}{2}y + \tfrac{1}{2}y^3)^{1/3} \\ &= \tfrac{1}{2}y + \tfrac{1}{4}y^2 + \tfrac{1}{24}y^3 + \tfrac{1}{24}y^4 + \tfrac{1}{48}y^5 + \tfrac{5}{288}y^6 \\ &\qquad + \tfrac{7}{576}y^7 + \tfrac{23}{2304}y^8 + \tfrac{329}{41472}y^9 + \tfrac{553}{82944}y^{10} + \ldots \end{aligned}$$

How are these related to random graphs on 2 or 3 vertices?

I have an analytic proof that $P_3(y) \succeq 0$, but it doesn't shed any light on the possible probabilistic interpretation.

Jim Fill has a probabilistic interpretation for $n = 2, 3$ in terms of birth-and-death chains, but it doesn't seem to generalize to $n \geq 4$.

A more general approach to the leading root $x_0(y)$ (When stumped, generalize . . . !)

Consider a *formal power series*

$$f(x, y) \; = \; \sum_{n=0}^{\infty} \alpha_n \, x^n \, y^{n(n-1)/2}$$

normalized to $\alpha_0 = \alpha_1 = 1$, or more generally

$$f(x, y) \; = \; \sum_{n=0}^{\infty} a_n(y) \, x^n$$

where

(a) $a_0(0) = a_1(0) = 1$;

(b) $a_n(0) = 0$ for $n \geq 2$; and

(c) $a_n(y) = O(y^{\nu_n})$ with $\lim_{n\to\infty} \nu_n = \infty$.

Coefficients can lie in an arbitrary commutative ring R.

In this general situation there is always a leading root $x_0(y)$, considered as a *formal power series*.

Examples:

- The "deformed exponential function"

$$F(x, y) \; = \; \sum_{n=0}^{\infty} \frac{x^n}{n!} \, y^{n(n-1)/2}$$

- The "partial theta function"

$$\Theta_0(x, y) \; = \; \sum_{n=0}^{\infty} x^n \, y^{n(n-1)/2}$$

- A q-series interpolation:

$$\widetilde{R}(x,y,q) \;=\; \sum_{n=0}^{\infty} \frac{x^n\, y^{n(n-1)/2}}{(1+q)(1+q+q^2)\,\cdots\,(1+q+\ldots+q^{n-1})}$$

For $q=0$ it reduces to the partial theta function.
For $q=1$ it reduces to the deformed exponential function.

- A simpler interpolation … the "deformed binomial series":

Start from the Taylor series for the binomial $f(x) = (1-\mu x)^{-1/\mu}$
[it is convenient to parametrize it in this way]
and introduce factors $y^{n(n-1)/2}$ as usual:

$$\begin{aligned} F_\mu(x,y) &= \sum_{n=0}^{\infty} (-\mu)^n \binom{-1/\mu}{n} x^n\, y^{n(n-1)/2} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{j=0}^{n-1} (1+j\mu) \right) x^n\, y^{n(n-1)/2} \end{aligned}$$

For $\mu = 0$ it reduces to the deformed exponential function.
For $\mu = 1$ it reduces to the partial theta function.

- Going farther … the "deformed hypergeometric series":

Note that exponential (${}_0F_0$) and binomial (${}_1F_0$) are simplest cases of the hypergeometric series ${}_pF_0$. We can apply "y-deformation" process to ${}_pF_0$:

$${}_pF_0^{*}\left(\begin{matrix} \mu_1, \ \ldots, \ \mu_p \\ \text{—} \end{matrix} \,\middle|\, x, y \right) \;=\; \sum_{n=0}^{\infty} (1;\mu_1)^{\overline{n}} \,\cdots\, (1;\mu_p)^{\overline{n}}\, \frac{x^n}{n!}\, y^{n(n-1)/2}$$

where

$$(1;\mu)^{\overline{n}} \;=\; \prod_{j=0}^{n-1} (1+j\mu)$$

Partial theta function Θ_0

Empirically $\xi_0(y)$ seems to have positive coefficients:

$$\begin{aligned}\xi_0(y) \;=\; & 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 \\ & +948y^9 + 2610y^{10} \;+\; \ldots \;+\; \text{terms through order } y^{6999}\end{aligned}$$

And $1/\xi_0(y)$ seems to have negative coefficients after the constant term 1:

$$\begin{aligned}\xi_0(y)^{-1} \;=\; & 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 \\ & -178y^9 - 490y^{10} \;-\; \ldots \;-\; \text{terms through order } y^{6999}\end{aligned}$$

Indeed, $1/\xi_0(y)^2$ seems to have this property:

$$\begin{aligned}\xi_0(y)^{-2} \;=\; & 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 \\ & -138y^9 - 386y^{10} \;-\; \ldots \;-\; \text{terms through order } y^{6999}\end{aligned}$$

Can this be proven?

Yes: By q-series identities.

The deformed binomial series

- Can prove (using explicit implicit function formula) that

$$\xi_0(y;\mu) \;=\; 1 + \sum_{n=1}^{\infty} \frac{P_n(\mu)}{d_n}\, y^n$$

  where $P_n(\mu)$ is a polynomial of degree n with integer coefficients and d_n are explicit integers.

- *Empirically* $P_n(\mu)$ has *two* interesting positivity properties:

  (a) $P_n(\mu)$ has all strictly positive coefficients.
  (b) $P_n(\mu) > 0$ for $\mu > -1$.

Can any of this be proven?

Alas, I have no idea … (except for the partial theta function, $\mu = 1$)

- This positivity even appears to extend to the deformed hypergeometric series ${}_pF_0^*\left(\begin{matrix}\mu_1, \;\ldots,\; \mu_p \\ \text{—}\end{matrix} \,\middle|\, x, y\right)$:

  the polynomials $P_n(\mu_1, \ldots, \mu_p)$ are coefficientwise positive jointly in the variables $\mu_1, \ldots, \mu_p$.

Why?

I wish I knew …

Joyeux anniversaire, Vincent!