Some wonderful conjectures (but very few theorems) at the boundary between analysis, combinatorics and probability

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Abstract. Many problems in combinatorics, statistical mechanics, number theory and analysis give rise to power series (whether formal or convergent) of the form

$$f(x,y) \;=\; \sum_{n=0}^{\infty} a_n(y)\, x^n \;,$$

where $\{a_n(y)\}$ are formal power series or analytic functions satisfying $a_n(0) \neq 0$ for $n = 0, 1$ and $a_n(0) = 0$ for $n \geq 2$. Furthermore, an important role is played in some of these problems by the roots $x_k(y)$ of $f(x,y)$ — especially the "leading root" $x_0(y)$, i.e. the root that is of order y^0 when $y \to 0$. Among the interesting series $f(x,y)$ of this type are the "partial theta function"

$$\Theta_0(x,y) \;=\; \sum_{n=0}^{\infty} x^n\, y^{n(n-1)/2} \;,$$

which arises in the theory of q-series, and the "deformed exponential function"

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2} \;,$$

which arises in the enumeration of connected graphs. These two functions can also be embedded in natural hypergeometric and q-hypergeometric families.

In this talk I will describe recent (and mostly unpublished) work concerning these problems — work that lies on the boundary between analysis, combinatorics and probability. In addition to explaining my (very few) theorems, I will also describe some amazing conjectures that I have verified numerically to high order but have not yet succeeded in proving. My hope is that one of you will succeed where I have not! Further information is available at `http://www.maths.qmul.ac.uk/~pjc/csgnotes/sokal`.