Quantum Quench of the nonlocal Luttinger Model

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Merci beaucoup VIncent, et joyeux anniversaire!

Plan of the talk

- Nonequilibrium dynamics of closed and isolated quantum many body system and quantum quench.
- The quenched Luttinger model.
- Rigorous Bosonization method.

Reference: Vieri Mastropietro , ZW “Quantum Quench for inhomogeneous states in the non-local Luttinger model ’ **Physical Reviews B, 91, 085123 (2015)**
Langmann Moosavi, Construction by Bosonization of a fermion-phonon model, **journal of mathematical physics, 56**, 091902 (2015)

The non-equilibrium physics

Consider a closed many body quantum system of size L that is isolated from the environment and reservoir; let the system prepared in a non-equilibrium initial state $|\Psi_0\rangle$ and time evolves w r t the Hamiltonian H: $|\Psi_t\rangle = e^{-iHt}|\Psi_0\rangle$. Then we consider the fate of the state $|\Psi\rangle = \lim_{t\to\infty} \lim_{L\to\infty} |\Psi_t\rangle$.

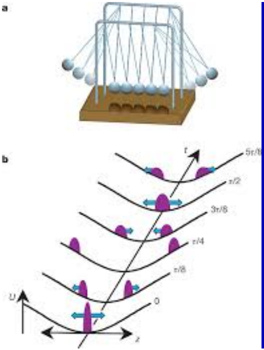
- Does $|\Psi\rangle$ evolves into any steady state or a stationary state;
- could we describe the state with thermodynamic ensemble;
- could we describe the general properties of such systems without going into the model detail; namely the universality.

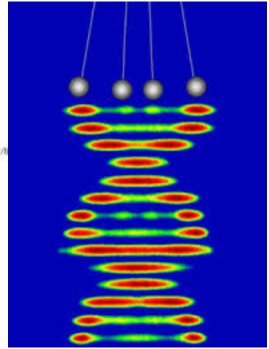
- these kind of questions are also called thermalizations of isolated systems, dated back to Von Neumann since 1939: only an academic question since the effect of environment could not be avoided;
- this question becomes interesting again due to the advances of cold atom experiments; especially the Bose Einstein condensation.
- One can prepare such system that is strongly correlated but very weakly coupled with the environment;

nonequilibrium thermalization

- nonequilibrium steady states exist; some are thermal states and some are not;
- Integrability plays very important role;

Quantum newton's cradle; T. Kinoshita, T. Wenger, and D.S. Weiss,Nature (London) 452, 854 (2008).





- Consider a system of about 1000 Rb87 atoms in a Harmonic trapping potential at very low temperature. the density of the Bose gas is prepared such that the interaction can be taken to be point like-2-body δ function interaction. Then the cloud is split into two counter-propagating clouds with opposite momentum, by the laser beam; the two clouds are then climbing up and down the harmonic potential by conservation of energy;

Quantum newton's cradle; T. Kinoshita, T. Wenger, and D.S. Weiss,Nature (London) 452, 854 (2008).

- The momentum distribution attains a stationary state for any spacial dimension, when t is big enough; when $d = 2, 3$ the system relax very quickly and thermalize. But in one dimension the system relaxes slowly to a non-thermal distribution;

Bose-Hubbard model, S. Trotzky Y.-A. Chen, A. Flesch, I. P. McCulloch, U. SchollwÃűck, J.Eisert, and I.Bloch, NaturePhys. 8, 325 (2012).

Consider a one dimensional Bosonic lattice system starting from a non-equilibrium state; the Hamiltonian is chosen to mimic the non-integrable Bose-Hubbard Hamiltonian:

$$H = \sum_j \left[-J(\hat{a}_j^+ a_{j+1} + h.c.) + U/2\hat{n}_j(\hat{n}_j - 1) + K/2\hat{n}_j j2 \right] \quad (-1)$$

- some observables approach stationary states that are compatible with thermal ones;
- the relaxation process is fast;
- more memory of the initial state
- more sensitive to approximations;

integrability plays important role for the non-equilibrium dynamics. It would be useful to have some mathematically rigorous results, as benchmark for experiments or further approximations;

One way to prepare the non equilibrium initial state for a cold atom system is by quantum quench. Namely we can prepare the system in a state that is an eigenstate of a Hamiltonian H_0, then we suddenly turn on or turn off some interactions by manipulating the laser beam or optical lattice so that the Hamiltonian becomes $H(\lambda) = H_0 + V(\lambda)$. Then we consider the time evolution w.r.t. $H(\lambda)$.

Luttinger model would be a good choice;

- 1) Better understand the non equilibrium phenomenons.
- 2) Test the techniques of equilibrium physics. We perform the calculation with both the Lieb Mattis method and Bosonization formula.

The nonlocal Luttinger Hamiltonian is

$$H_\lambda = \int_{-L/2}^{L/2} dx \; -i(:\psi^+_{x,1}\partial_x\psi^-_{x,1}: - :\psi^+_{x,2}\partial_x\psi^-_{x,2}:) + \lambda\int_{-\frac{L}{2}}^{\frac{L}{2}} dxdyv(x-y):\psi^+_{x,1}\psi^-_{x,1}::\psi^+_{y,2}\psi^-_{y,2}: \quad (-2)$$

where

- $\psi^\pm_{x,\omega} = \frac{1}{\sqrt{L}}\sum_k a_{k,\omega}e^{\pm ikx}$, $\omega = 1,2$, $k = \frac{2\pi n}{L}$ with $n \in N$ are fermionic creation or annihilation operators,
- $v(x)$ is a smooth short range potential such that $|v(x-y)| \leq e^{-\alpha|x-y|^2}$ or $|v(x-y)| \leq \frac{1}{(1+|x-y|^\alpha)}$ for $\alpha > 1$.
- Difference from the local Luttinger model for which $v(x-y) = \delta(x-y)$. This causes the ultraviolet divergence, as the Fourier transform doesn't decay for large momentum.

The Hamiltonian can be rewritten as

$$H_\lambda = H_0 + V = \sum_{k>0} k[(a^+_{k,1} a^-_{k,1} + a^-_{-k,1} a^+_{-k,1}) + (a^+_{-k,2} a^-_{-k,2} + a^-_{k,2} a^+_{k,2})$$
$$+ \frac{2\lambda}{L} \sum_{p>0} [\rho_1(p)\rho_2(-p) + \rho_1(-p)\rho_2(p)] + \frac{\lambda}{L} \hat{v}(0) N_1 N_2 \quad (-3)$$

where, if $p > 0$,

$$\rho_\omega(p) = \sum_k a^+_{k+p,\omega} a^-_{k,\omega}$$

$$N_1 = \sum_{k>0} (a^+_{k,1} a^-_{k,1} - a^-_{-k,1} a^+_{-k,1}) \quad N_2 = \sum_{k>0} (a^+_{-k,2} a^-_{-k,2} - a^-_{k,2} a^+_{k,2})$$

The regularization implicit in the above expressions is that $\rho_\omega(p)$ must be thought as $\lim_{\Lambda\to\iota} \sum_k \chi_\Lambda(k)\chi_\Lambda(k+p) a^+_{k+p,\omega} a^-_{k,\omega}$;

The basic property of the Luttinger model is the validity of the following anomalous commutation relations,

$$[\rho_1(-p), \rho_1(p')] = \frac{pL}{2\pi}\delta_{p,p'} \quad [\rho_2(-p), \rho_2(p')] = -\frac{pL}{2\pi}\delta_{p,p'}. \tag{-4}$$

Let $|0>$ be the ground state of H_0, then we have:

$$\rho_2(p)|0>= 0, \quad \rho_1(-p)|0>= 0. \tag{-5}$$

Introducing the operator $T = \frac{1}{L}\sum_{p>0}[\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)]$ and write $H = (H_0 - T) + (V + T) = H_1 + H_2$.
H_2 can be diagonalized by the following transformation:

$$\begin{aligned} & e^{iS} H_2 e^{-iS} = \tilde{H}_2 \\ & = \frac{2\pi}{L}\sum_p \mathrm{sech} 2\phi(p)[\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)] + E_0, \end{aligned} \quad (-6)$$

where

$$S = \frac{2\pi}{L}\sum_{p\neq 0}\phi(p)p^{-1}\rho_1(p)\rho_2(-p), \quad \tanh\phi(p) = -\frac{\lambda v(p)}{2\pi}. \quad (-7)$$

Luttinger model and the Lieb-Mattis solution

Define

$$D = \tilde{H}_2 - T = \frac{2\pi}{L} \sum_p \sigma(p)[\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)] + E_0, \qquad (-8)$$

$\sigma(p) = \mathrm{sech} 2\phi(p) - 1$ and we have $[H_0, D] = 0$. and the diagonalization formula for the Hamiltonian reads:

$$e^{iS} e^{iHt} e^{-iS} = e^{i(H_0+D)t}, \qquad (-9)$$

The average of the two-point function over the ground state reads:

$$< GS|\psi^+_{\omega,x}\psi^-_{\omega,0}|GS >= \frac{1}{2\pi} \frac{1}{-i\epsilon_\omega x + 0^+} \exp \int_0^\infty dp \frac{1}{p} \left[2 \sinh^2 \phi_p (\cos px - 1)\right]. \quad (-10)$$

Asymptotically, for large distances

$$< GS|\psi^+_{\omega,x}\psi^-_{\omega,0}|GS >\sim O(|x|^{-1-\eta}) \qquad (-11)$$

implying that the average of the occupation number over the interacting state is $n_{k'+\epsilon_\omega p_F} \sim a + O(k'^\eta)$, where a is a nonnegative real number.

For obtaining the above result one should frequently use:

$$e^{i\varepsilon S}\psi^-_{1,x}e^{-i\varepsilon S} = W_{1,x}R_{1,x}\psi^-_{1,x} \tag{-12}$$

with $c(\phi) = \cosh\varepsilon\phi - 1$, $s(\phi) = \sinh\varepsilon\phi$

$$W^{\varepsilon}_{1,x} =$$
$$\exp\{-\frac{2\pi}{L}\sum_{p>0}\frac{e^{-0^+p}}{p}[\rho_1(-p)e^{ipx} - \rho_1(p)e^{-ipx}]c(\phi)\}$$
$$R^{\varepsilon}_{1,x} =$$
$$\exp\{-\frac{2\pi}{L}\sum_{p>0}\frac{e^{-0^+p}}{p}[\rho_2(-p)e^{ipx} - \rho_2(p)e^{-ipx}]s(\phi)\}$$

Quench dynamics of the ground states, M Cazalilla *Phys. Rev. Lett.* 97 156403 (2006), Rentrop, J.; Schuricht, D.; Meden, V. *New J. Of Phys.* 14 075001, (2012)

Let $|O_t >= e^{iH_\lambda t}|0>$, $|0>$ is the ground state of H_0 but not an eigenstate of H; ($|GS>= e^{iS}|0>$) We have

$$< O_t|\psi^+_{\omega,x}\psi^-_{\omega,y}|O_t >_\lambda=< 0|e^{-iH_\lambda t}\psi^+_{\omega,x}\psi^-_{\omega,y}e^{iH_\lambda t}|0>, \tag{-13}$$

and we obtain

$$< O_t|\psi^+_{\omega,x}\psi^-_{\omega,0}|O_t >= \frac{i}{2\pi}\frac{1}{\varepsilon_\omega x + i0^+} \tag{-14}$$
$$\exp\int_0^\iota dp\frac{\gamma(p)}{p}\{(\cos px - 1)(1 - \cos 2p(\sigma_p + 1)t)\},$$

where $\gamma(p) = 4\sinh^2\phi_p\cosh^2\phi_p$.

$$\lim_{t\to\infty} < O_t|\psi^+_{\omega,x}\psi^-_{\omega,0}|O_t >= \frac{i}{2\pi}\frac{1}{\varepsilon_\omega x + i0^+}$$
$$\exp\int_0^\infty dp\frac{1}{p}\{\gamma(p)(\cos\ px - 1)\} \tag{-15}$$

Evolution of inhomogeneous initial state

let

$$|I_t >= e^{iH_\lambda t}(e^{ip_F x}\psi^+_{1,x} + e^{-ip_F x}\psi^+_{2,x})|0> . \qquad (-16)$$

$n(z)$ is the regularized particle number

$$n(z) = \frac{1}{2}\sum_{\rho=\pm}(\tilde{\psi}^+_{1,z+\rho\varepsilon}\tilde{\psi}^-_{2,z} + \tilde{\psi}^+_{2,z+\rho\varepsilon}\psi^-_{1,z} + \psi^+_{2,z+\rho\varepsilon}\tilde{\psi}^-_{2,z} + \tilde{\psi}^+_{1,z+\rho\varepsilon}\tilde{\psi}^-_{1,z}) \qquad (-17)$$

$< I_{\lambda,t}|n(z)|I_{\lambda,t} >$ is sum of several terms

$$\begin{aligned}
&\langle 0|\tilde{\psi}^-_{1,x}e^{iHt}\tilde{\psi}^+_{1,z+\rho\varepsilon}\tilde{\psi}^-_{2,z}e^{-iHt}\tilde{\psi}^+_{2,x}\,|0\rangle + \\
&\langle 0|\,\tilde{\psi}^-_{2,x}e^{iHt}\tilde{\psi}^+_{2,z+\rho\varepsilon}\psi^-_{1,z}e^{-iHt}\psi^+_{1,x}\,|0\rangle + \\
&\langle 0|\,\tilde{\psi}^-_{1,x}e^{iHt}\tilde{\psi}^+_{2,z+\rho\varepsilon}\tilde{\psi}^-_{2,z}e^{-iHt}\tilde{\psi}^+_{1,x}\,|0\rangle + \\
&\langle 0|\,\tilde{\psi}^-_{2,x}e^{iHt}\tilde{\psi}^+_{1,z+\rho\varepsilon}\tilde{\psi}^-_{1,z}e^{-iHt}\tilde{\psi}^+_{2,x}\,|0\rangle + \\
&\langle 0|\,\tilde{\psi}^-_{1,x}e^{iHt}\tilde{\psi}^+_{1,z+\rho\varepsilon}\tilde{\psi}^-_{1,z}e^{-iHt}\tilde{\psi}^+_{1,x}\,|0\rangle + \\
&\langle 0|\,\tilde{\psi}^-_{2,x}e^{iHt}\tilde{\psi}^+_{2,z+\rho\varepsilon}\tilde{\psi}^-_{2,z}e^{-iHt}\tilde{\psi}^+_{2,x}\,|0\rangle .
\end{aligned}$$

$$\lim_{\varepsilon,\delta\to 0, L\to\infty} < I_t|n(z)|I_t >_\lambda = \frac{1}{4\pi^2}\ [\frac{1}{((x-z)-t)^2} + \frac{1}{((x-z)-t)^2}] + \quad (-18)$$
$$\frac{1}{4\pi^2}\ \frac{e^{Z(t)}}{(x-z)^2-t^2}[\ e^{2ip_F(x-z)}Q_a(x,z,t) + e^{-2ip_F(x-z)}Q_b(x,z,t)\ ],$$

where

$$Z(t) = \int_0^\infty \frac{dp}{p}\gamma(p)(\cos 2p(\sigma+1)t - 1), \quad \gamma(p) = \frac{e^{4\phi(p)}-1}{2} \quad (-19)$$

and

$$\begin{aligned} Q_a = \quad & + \quad \exp\{\int_0^\iota dp\frac{1}{p}[(e^{ip(x-z)+ip(\sigma(p)+1)t-0^+(\sigma+1)p} - e^{ip(x-z)+ipt-0^+p}) \\ & + \quad (e^{ip(x-z)-ip(\sigma(p)+1)t-0^+(\sigma+1)p} - e^{ip(x-z)-ipt-0^+p})]\} \quad (-20) \end{aligned}$$

$$\begin{aligned} Q_b = \quad & + \quad \exp\{\int_0^\iota dp\frac{1}{p}[(e^{-ip(x-z)+ip(\sigma+1)t-0^+(\sigma(p)+1)p} - e^{-ip(x-z)+ipt-0^+p}) \\ & + \quad (e^{-ip(x-z)-ip(\sigma(p)+1)t-0^+(\sigma+1)p} - e^{-ip(x-z)-ipt-0^+p})]\}. \quad (-21) \end{aligned}$$

If we replace σ_p with σ_0 we have

$$\frac{1}{(x-z)^2 - t^2} e^{Q_a} = \frac{1}{(x-z)^2 - (1+\sigma_0)^2 t^2}, \quad (-22)$$

For large t we have

$$\exp Z(t) = O(t^{-\gamma(0)}), \quad (-23)$$

the bosonization formula

Certain two dimensional model can be described equivalently as a Bosonic model or Fermionic model.

- $\mathcal{H}_{\mathcal{B}} \sim \mathcal{H}_{\mathcal{F}}$;
- $H_B = H_F + 1/2[(Q_1)^2 + (Q_2)^2]$,
- exact bosonization formula

The Bosonization formula

The exact Bosonization formula for the Fermionic fields are given by (Langmann, Moosavi, J. Math. Phys. (2015), R. Heidenreich, R. Seiler, and D. A. Uhlenbrock, J. Stat. Phys. (1980).)

$$\begin{aligned}\psi_\omega(x) &= :\lim_{\delta\to 0} N_\delta e^{i\pi\varepsilon_\omega x Q_\omega/L} R_\omega^{-\varepsilon_\omega} e^{i\pi\varepsilon_\omega x Q_\omega/L} \exp\{\varepsilon_\omega \sum_{p>0} \frac{2\pi}{Lp}[\rho_\omega(p)e^{-ipx-\delta|p|} \\ &- \rho_\omega(-p)e^{ipx-\delta|p|}]\}, \\ \psi_\omega^\dagger(x) &= :\lim_{\delta\to 0} N_\delta e^{-i\pi\varepsilon_\omega x Q_\omega/L} R_\omega^{\varepsilon_\omega} e^{-i\pi\varepsilon_\omega x Q_\omega/L} \exp\{-\varepsilon_\omega \sum_{p>0} \frac{2\pi}{Lp}[\rho_\omega(p)e^{-ipx-\delta|p|} \\ &- \rho_\omega(-p)e^{ipx-\delta|p|}]\},\end{aligned}$$

$\omega, \omega' = 1, 2$, $\varepsilon_1 = +$, $\varepsilon_2 = -$, $Q_\omega = \rho_\omega(0)$ and $N_\delta = \left[\frac{1}{L(1-e^{-2\pi\delta/L})}\right]^{1/2}$ is a numerical constant depending on the regularization parameter δ; $R_\omega^\pm$ are the Klein factors such that $R_\omega^- = (R_\omega)^\dagger = (R_\omega^+)^\dagger$. $[\rho_\omega(p), R_{\omega'}] = \varepsilon_\omega \delta_{\omega,\omega'} \delta_{p,0} R_\omega$

- Loop groups $\mathcal{G} = \mathrm{Map}(S^1_L, G)$ are groups of continuous maps from a circle S^1_L into a certain compact Lie group G. consider the case $G = U(1)$.
- Let f a real valued function on the interval $[-L/2, L/2]$ and e^{if} a periodic function on the same interval. Then the loop group $\mathcal{G}$ can be considered as the set of periodic functions $\{e^{if}\}$ with pointwise production $(e^{if_1})(y) \cdot (e^{if_2})(y) = e^{i(f_1+f_2)(y)} = e^{i[f_1(y)+f_2(y)]}$.

All these things are very well known and here are the basic references:

- G. Segal, Commun. Math. Phys. 80, 301-342 (1981).
- I. B. Frenkel and V. G. Kac, Invent. Math. 62, 23-66 (1980).
- A. L. Carey and C. A. Hurst, Commun. Math. Phys. 98, 435-448 (1985).
- A. L. Carey and S. N. M. Ruijsenaars, On fermion gauge groups, current algebras and Kac-Moody algebras, Acta Appl. Math. 10, 1-86 (1987).
- A. L. Carey and E. Langmann, Loop groups and quantum fields, Progress in Mathematics, Vol. 205, (Birkhauser, 2002), pp. 45-94.

It is well known that $\mathcal{G}$ has an interesting central extension $\hat{\mathcal{G}} = U(1) \times \mathcal{G}$ with the group multiplication

$$(g_1, \phi_1) \cdot (g_2, \phi_2) = (g_1 g_2 \sigma(\phi_1, \phi_2), \phi_1 \cdot \phi_2),\ g_i \in U(1),\ \phi_i = e^{if_i}, \tag{-24}$$

where

$$\sigma(\phi_1, \phi_2) = \sigma(e^{if_1}, e^{if_2}) = e^{-iS(f_1, f_2)/2} \tag{-25}$$

is a two cocycle of $\mathcal{G}$ and the explicit form of $S(f_1, f_2)$ reads (G. Segal, CMP (1981).):

$$\begin{aligned} S(f_1, f_2) &= \frac{1}{4\pi}[f_1(L/2)f_2(-L/2) - f_1(-L/2)f_2(L/2)] \\ &+ \frac{1}{4\pi}\int_{-L/2}^{L/2} dx\left[\frac{df_1(x)}{dx}f_2(x) - f_1(x)\frac{df_2(x)}{dx}\right]. \end{aligned} \tag{-26}$$

consider the projective representation of $\mathcal{G} = \mathrm{Map}(S^1_L, G)$ with $G = U(1)$ realized by a unitary operator $\Gamma_\omega(\phi)$, $\omega = 1, 2$ on the Fermi Fock space. We have

$$\Gamma(\phi_1)\Gamma(\phi_2) = \sigma(\phi_1, \phi_2)\Gamma(\phi_1 \cdot \phi_2), \text{ and } \Gamma(\phi)^* = \Gamma(\phi^*). \tag{-27}$$

We can choose the function f as:

$$f(y) = n\frac{2\pi}{L}y + \alpha(y), \quad y \in [-L/2, L/2], \tag{-28}$$

where $\alpha(y) = \alpha(y + L)$ is a periodic function and $n \equiv [f(L/2) - f(-L/2)]/2\pi \in \mathbb{Z}$, is called the winding number. so e^{if} is periodic function;
We can decompose $\alpha(y)$ into Fourier modes

$$\alpha(y) = \sum_p \frac{2\pi}{L}\hat{\alpha}(p)e^{-ipy} \tag{-29}$$

We can further decompose the Fourier modes as

$$\hat{\alpha}(p) = \hat{\alpha}^+(p) + \hat{\alpha}^-(p) + \hat{\alpha}(0). \tag{-30}$$

For the periodic function e^{if} chosen as above we can define the operators $\Gamma_\omega(e^{if})$ as:

$$\Gamma_\omega(e^{if})(y) \equiv e^{i\varepsilon_\omega \alpha(0) Q_\omega \frac{\pi}{L} y} R_\omega^{-\varepsilon_\omega} e^{i\varepsilon_\omega \alpha(0) Q_\omega \frac{\pi}{L} y} \exp\left\{\varepsilon_\omega \sum_{p\neq 0} \frac{2\pi}{L} \alpha(p)\rho_\omega(p) e^{-ipy}\right\}, \tag{-31}$$

Then we define

$$\begin{aligned} & \psi(y,\delta) = \Gamma_\omega(e^{if})(y,\delta) \\ \equiv\quad & N_\delta e^{i\varepsilon_\omega \alpha(0) Q_\omega \frac{\pi}{L} y} R_\omega^{-\varepsilon_\omega} e^{i\varepsilon_\omega \alpha(0) Q_\omega \frac{\pi}{L} y} \exp\left\{\varepsilon_\omega \sum_{p\neq 0} \frac{2\pi}{L} \alpha(p)\rho_\omega(p) e^{-ipy-\delta p}\right\}, \end{aligned}$$

The Klein factors and the chiral charge factors can be expressed as the operators realizing the projective representations as follows:

$$R_\omega := \Gamma_\omega(e^{i\frac{2\pi}{L}y}) \quad \text{and} \quad e^{i\hat{\alpha}(0)Q_\omega} := \Gamma_\omega(e^{i\hat{\alpha}(0)}). \tag{-32}$$

We have the following commutation relation for the Klein factors and the charge operators.

$$R_\omega^{-1} Q_{\omega'} R_\omega = Q_{\omega'} + I\delta_{\omega\omega'}, \tag{-33}$$

where I is the identity operator.

Proof.

We consider only the nontrivial case such that $\omega = \omega'$ and we forget this index in the rest of this section. Since $\hat{\alpha}(0)$ is a real number, the set of operators $e^{i\hat{\alpha}(0)Q_\omega}$ forms a one parameter group with the parameter $\hat{\alpha}(0)$ and we have:

$$iR^{-1}QR = \frac{d}{d\hat{\alpha}(0)}[R^{-1}e^{i\hat{\alpha}(0)Q}R]|_{\hat{\alpha}(0)=0}. \quad (-34)$$

On the other hand we have

$$\begin{aligned} R^{-1}e^{i\hat{\alpha}(0)Q}R &= \Gamma(e^{-i\frac{2\pi}{L}y})\Gamma(e^{i\hat{\alpha}(0)})\Gamma(e^{i\frac{2\pi}{L}y}) \\ &= e^{-iS(-\frac{2\pi}{L}y,\ \hat{\alpha}(0))/2}\Gamma(e^{-i\frac{2\pi}{L}y+i\hat{\alpha}(0)})\Gamma(e^{i\frac{2\pi}{L}y}) \\ &= e^{-iS(-\frac{2\pi}{L}y,\ \hat{\alpha}(0))/2}e^{-iS(-\frac{2\pi}{L}y+\hat{\alpha}(0),\ \frac{2\pi}{L}y)/2}\Gamma(e^{-i\frac{2\pi}{L}y+i\hat{\alpha}(0)+i\frac{2\pi}{L}y}). \end{aligned} \quad (-35)$$

Using the explicit formula for the cocycle (-26) we have

$$e^{-iS(-\frac{2\pi}{L}y,\ \hat{\alpha}(0))/2} = e^{i\hat{\alpha}(0))/2},\ e^{-iS(-\frac{2\pi}{L}y+\hat{\alpha}(0),\ \frac{2\pi}{L}y)/2} = e^{i\hat{\alpha}(0))/2} \quad (-36)$$

So we have

$$\frac{d}{d\hat{\alpha}(0)}[R^{-1}e^{i\hat{\alpha}(0)Q}R]|_{\hat{\alpha}(0)=0} = \frac{d}{d\hat{\alpha}(0)}[e^{i\hat{\alpha}(0)Q}e^{i\hat{\alpha}(0)}]|_{\hat{\alpha}(0)=0} = i(Q+I). \quad (-37)$$